Elliptic Equations Involving Measures

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1 Introduction

The role of measures in the study of nonlinear partial differential equations has became more and more important in the last years, not only because it belongs to the mathematical spirit to try to extend the scope of a theory, but also because the extension from the function setting to the measure framework appeared to be the only way to bring into light nonlinear phenomena and to explain them. In a very similar process, the theory of linear equations shifted from the function setting to the distribution framework. The aim of this chapter is to bring into light several aspects of this interaction, in particular its connection with the singularity theory and the nonlinear trace theory. Our intention is not to present a truly self-contained text : clearly we shall assume that the reader is familiar with the standard second order linear elliptic equations regularity theory, as it is explained in Gilbarg and Trudinger's classical treatise [47]. Part of the results will be fully proven, and, for some of them, only the statements will be exposed. The starting point is the linear theory, in our case the study of

$$\begin{aligned} Lu &= \lambda \quad \text{in } \Omega, \\ u &= \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, L is a linear elliptic operator of second order, and λ and μ are Radon measures, respectively in Ω and $\partial\Omega$. Under some structural and regularity assumptions on L (essentially that the maximum principle holds), it is proven that (1.1) admits a unique solution. Moreover this solution admits a linear representation, i.e.

$$u(x) = \int_\Omega G_L^\Omega(x,y)d\lambda(y) + \int_{\partial\Omega} P_L^\Omega(x,y)d\mu(y), \tag{1.2}$$

for any $x \in \Omega$, where G_L^Ω and P_L^Ω are respectively the Green and the Poisson kernels associated to L in Ω. The presentation that we adopt is a combination of the classical regularity theory for linear elliptic equations and Stampacchia duality approach which provides the most powerful tool for the extension to semilinear equations. In Section 3 we shall concentrate on semilinear equations with an absorption-reaction term of the following type

$$\begin{aligned} Lu + g(x,u) &= \lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $(x,r) \mapsto g(x,r)$ is a continuous function defined in $\Omega \times \mathbb{R}$, satisfying the absorption principle

$$\operatorname{sign}(r) g(x,r) \geq 0, \quad \forall (x,r) \in \Omega \times (-\infty, -r_0] \cup [r_0, \infty), \tag{1.4}$$

for some $r_0 \geq 0$. Under general assumptions on g, which are the natural generalisation of the Brezis-Bénilan weak-singularity condition [11], it is proven that for any Radon measure λ in Ω satisfying

$$\int_\Omega \rho_{\partial\Omega}^\alpha d|\lambda| < \infty, \tag{1.5}$$

with $\rho_{\partial\Omega}(x) = \operatorname{dist}(x, \partial\Omega)$ and $\alpha \in [0,1]$, Problem (1.3) admits a solution. Notice that the assumption on g depends both on n and α. Furthermore, uniqueness holds if $r \mapsto g(x,r)$ is nondecreasing, for any $x \in \Omega$. However, the growth condition on g is very restrictive. Thus the problem may not be solved for all the measures, but only for specific ones. A natural condition is to assume that the measure λ satisfies

$$\int_\Omega g\left(x, \mathbb{G}_L^\Omega(|\lambda|)\right) \rho_{\partial\Omega} dx < \infty, \tag{1.6}$$

where $\mathbb{G}_L^\Omega(|\lambda|)$, defined by

$$\mathbb{G}_L^\Omega(|\lambda|)(x) = \int_\Omega G_L^\Omega(x,y) d|\lambda|(y), \quad \forall x \in \Omega,$$

is called the Green potential of $|\lambda|$. Under an additional condition on g, called the Δ_2-condition, which excludes the exponential function, but not any positive power, it is shown that, in Condition (1.6), the measure λ can be replaced by its singular part with respect to the n-dimensional Hausdorff measure in the Lebesgue decomposition, in order Problem (1.3) to be solvable. In the case where

$$g(x,r) = |r|^{q-1} r,$$

with $r > 0$, Problem (1.3) can be solved for any bounded measure if $0 < q < n/(n-2)$, but this is no longer the case if $q \geq n/(n-2)$. Baras and Pierre provide in [9] a necessary and sufficient condition on the measure λ in terms of Bessel capacities. The solvability of nonlinear equations with measure is closely associated to removability question, the standard one being the following : assume K is a compact subset of Ω and u a solution of

$$Lu + g(x,u) = 0 \quad \text{in } \Omega \setminus K, \tag{1.7}$$

does it follows that u can be extended, in a natural way, so that the equation is satisfied in all Ω? The answer is positive if some Bessel capacity, connected to the growth of g, of the set K is zero. In Section 4 we give an overview of the semilinear problem with a source-reaction term of the following type

$$\begin{aligned} &Lu = g(x,u) + \lambda && \text{in } \Omega, \\ &u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.8}$$

For this equation, not only the concentration of the measure is important, but also the total mass. The first approach, due to Lions [66] is to construct a supersolution, the conditions are somehow restrictive. In the convex case, a rather complete presentation is provided by Baras and Pierre [10], with the improvement of Adams and Pierre [2]. The idea is to write the solution u of (1.8) under the form

$$u(x) = \int_\Omega G_L^\Omega g(y,u(y))dy + \mathbb{G}_L^\Omega(\lambda) \quad \text{in } \Omega. \tag{1.9}$$

The convexity of $r \mapsto g(x,r)$ gives a necessary condition expressed in term of the conjugate function $g^*(x,r)$. The difficulty is to prove that this condition is also sufficient and to link it to a functional analysis framework. An extension of this method is given by Kalton and Verbitsky [52] in connection with weighted inequalities in L^q spaces. Finally, conditions for removability of singularities of positive solutions are treated by Baras and Pierre [9]. In Section 5 we consider the problem of solving boundary value problems with measures data for nonlinear equations with an absorption-reaction term,

$$\begin{aligned} Lu + g(x,u) &= 0 && \text{in } \Omega, \\ u &= \mu && \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

The first results in that direction are due to Gmira and Véron [48] who prove that the Bénilan-Brezis method can be adapted in a framework of weighted Marcinkiewicz spaces for obtaining existence of solutions in the so-called subcritical case : the case in which the problem is solvable with any boundary Radon measure. In a similar way as for Problem (1.3), it is shown that Problem (1.10) is solvable if the measure μ satisfies

$$\int_\Omega g\left(x, \mathbb{P}_L^\Omega(|\mu|)\right) \rho_{\partial\Omega} dx < \infty, \tag{1.11}$$

where

$$\mathbb{P}_L^\Omega(|\mu|)(x) = \int_{\partial\Omega} P_L^\Omega(x,y) d\,|\mu|\,(y), \quad \forall x \in \Omega.$$

It is also possible to extend the range of solvability if μ is replaced by its singular part with respect to the $(n-1)$-dimensional Hausdorff measure, for specific functions g which verify a power like growth. In the last years the model case of equation

$$Lu + |u|^{q-1} u = 0, \tag{1.12}$$

acquired a central role because of its applications. The case $q = (n+2)/(n-2)$ is classical in Riemannian geometry and corresponds to conformal change of metric with prescribed constant negative scalar curvature [67], [87]. The case $1 < q \leq 2$ is associated to superprocess in probability theory. It has been developed by Dynkin, [34], [35] and Le Gall [62] who introduced very powerful new tools for studying the properties of the positive solutions of this equation. The central idea is the discovery by Le Gall [61], in the case $q = 2 = n$, and the extension by Marcus and Véron [68], in the general case $q > 1$ and $n \geq 2$, of the existence of a boundary trace of positive solutions of (1.12) in a smooth bounded domain Ω. This boundary trace denoted by $Tr_{\partial\Omega}(u)$ is no longer a Radon measure, but a σ-finite Borel measure which can takes infinite value on compact subsets of the boundary. The critical value for this equation, first observed by Gmira and Véron, is $q_c = (n+1)/(n-1)$. It is proven in [61], [70] that for any positive σ-finite Borel measure μ on $\partial\Omega$ the problem

$$\begin{aligned} Lu + |u|^{q-1} u &= 0 \quad \text{in } \Omega, \\ Tr_{\partial\Omega}(u) &= \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{1.13}$$

admits a unique solution provided $1 < q < q_c$. This is no longer the case when $q \geq q_c$. Although many results are now available for solving the super-critical case of Problem (1.13), the full theory is not yet completed. An important colateral problem deals with the question of boundary singularities, an example of which is the following : suppose K is a compact subset of $\partial\Omega$, $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus K)$ is a solution of (1.13) in Ω which vanishes on $\partial\Omega \setminus K$; does it imply that u is identically zero ? The answer to this question is complete, and expressed in terms of boundary Bessel capacities.

2 Linear equations

2.1 Elliptic equations in divergence form

We call $x = (x_1, \ldots, x_n$ the variables in the space $\mathbb{R}^n$. Let Ω be a bounded domain in $\mathbb{R}^n$. The type of operators under consideration are linear second order differential operators in divergence form

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(c_i u) + du \tag{2.1}$$

where the a_{ij}, b_i, c_i and d are at least bounded measurable functions satisfying the uniform ellipticity condition in Ω :

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \tag{2.2}$$

for almost all $x \in \Omega$, where $\alpha > 0$ is some fixed constant. It is classical to associate to L the bilinear form A_L

$$A_L(u,v) = \int_\Omega a_L(u,v)dx, \quad \forall u,v \in W_0^{1,2}(\Omega), \tag{2.3}$$

where

$$a_L(u,v) = \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^n \left(b_i \frac{\partial u}{\partial x_i} v + c_i \frac{\partial v}{\partial x_i} u \right) + duv. \tag{2.4}$$

An important uniqueness condition, symmetric in the b_i and c_i, which also implies the maximum principle, is the following :

$$\int_\Omega \left(dv + \sum_{i=1}^n \frac{1}{2}(b_i + c_i) \frac{\partial v}{\partial x_i} \right) dx \geq 0, \quad \forall v \in C_c^1(\Omega), v \geq 0. \tag{2.5}$$

Lemma 2.1 *Let the coefficients of L be bounded and measurable, and conditions (2.2) and (2.5) hold. Then for any $\phi \in W^{1,2}(\Omega)$ and $f_i \in L^2(\Omega)$ ($i = 0, \ldots, n$) there exists a unique $u \in W^{1,2}(\Omega)$ solution of*

$$\begin{aligned} Lu &= f_0 - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} && \text{in } \Omega, \\ u &= \phi && \text{on } \partial\Omega, \end{aligned} \tag{2.6}$$

Proof. By a solution, we mean $u - \phi \in W_0^{1,2}(\Omega)$ and

$$A_L(u,v) = \int_\Omega \left(f_0 v + \sum_{i=1}^n f_i \frac{\partial v}{\partial x_i} \right) dx, \quad \forall v \in W_0^{1,2}(\Omega). \tag{2.7}$$

We put $\tilde{u} = u - \phi$. Then solving (2.6) is equivalent to finding $\tilde{u} \in W_0^{1,2}(\Omega)$ such that

$$A_L(\tilde{u},v) = \int_\Omega \left(f_0 v + \sum_{i=1}^n f_i \frac{\partial v}{\partial x_i} - a_L(\phi,v) \right) dx, \quad \forall v \in W_0^{1,2}(\Omega). \tag{2.8}$$

The bilinear form A_L is clearly continuous on $W_0^{1,2}(\Omega)$ and

$$A_L(v,v) = \int_\Omega \left(\sum_{i,j=1}^n a_{ij} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} + dv^2 + \frac{1}{2} \sum_{i=1}^n (b_i + c_i) \frac{\partial v^2}{\partial x_i} \right) dx.$$

By (2.2) and (2.5),

$$A_L(v,v) \geq \alpha \int_\Omega |\nabla v|^2 \, dx, \quad \forall v \in C_0^1(\Omega).$$

By density A_L is coercive and thanks to Lax-Milgram's theorem, it defines an isomorphism between the Sobolev space $W_0^{1,2}(\Omega)$ and its dual space $W^{-1,2}(\Omega)$. $\square$

The celebrated De Giorgi-Nash-Moser regularity theorem asserts that, for $p > n$ and $f \in L^p_{loc}(\Omega)$, any $W^{1,2}_{loc}(\Omega)$ function u which satisfies

$$\int_\Omega a_L(u,\phi)dx = \int_\Omega f\phi dx, \quad \forall \phi \in C_0^\infty(\Omega), \tag{2.9}$$

is locally Hölder continuous, up to a modification on a set of measure zero. Furthermore the weak maximum principle holds in the sense that if $u \in W^{1,2}(\Omega)$ satisfies

$$A_L(u,\phi) \le 0, \quad \forall \phi \in C_0^\infty(\Omega),\ \phi \ge 0, \tag{2.10}$$

such a u is called a weak sub-solution, there holds

$$\sup_\Omega u \le \sup_{\partial\Omega} u. \tag{2.11}$$

In the above formula,

$$\sup_{\partial\Omega} v := \inf\{k \in \mathbb{R} :\ (v-k)_+ \in W_0^{1,2}(\Omega)\}.$$

At end, the strong maximum principle holds : if for some ball $B \subset \bar{B} \subset \Omega$,

$$\sup_B u = \sup_\Omega u, \tag{2.12}$$

then u is constant in the connected component of Ω containing B.

If the a_{ij} and the c_i are Lipschitz continuous, and the b_i and d are bounded measurable functions, the operator L can be written in non-devergence form

$$Lu = -\sum_{i,j=1}^n a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^n b'_j \frac{\partial u}{\partial x_j} + d'u, \tag{2.13}$$

where

$$b'_j = b_j - c_j - \sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_i}, \quad d' = d - \sum_{i=1}^n \frac{\partial c_i}{\partial x_i}.$$

Conversely, an operator L in the non-divergence form (2.13) with Lipschitz continuous coefficients a_{ij} and bounded and measurable coefficients b'_i can be written in divergence form

$$Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{j=1}^n \tilde{b}_j \frac{\partial u}{\partial x_j} + \tilde{d}, \tag{2.14}$$

with

$$\tilde{b}_j = b'_j + \sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_i}.$$

This duality between operators in divergence or in non-divergence form is very useful in the applications, in particular in the regularity theory of solutions of elliptic equations. If L is defined by (2.1), the adjoint operator L^* is defined by

$$L^*\phi = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \phi}{\partial x_i}\right) + \sum_{i=1}^{n} c_i \frac{\partial \phi}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i\phi) + d\phi. \tag{2.15}$$

Under the mere assumptions that the coefficients a_{ij}, b_i, c_i and d are bounded and measurable in Ω, the uniform ellipticity (2.2), and the uniqueness condition (2.5), the two operators L and L^* define an isomorphism between $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$. If the a_{ij} and the b_i are Lipschitz continuous, for any $u \in L^1_{loc}(\Omega)$, Lu can be considered as a distribution in Ω if we define its action on test functions in the following way :

$$\langle Lu, \phi\rangle = \int_\Omega uL^*\phi dx, \quad \forall \phi \in C_0^\infty(\Omega). \tag{2.16}$$

2.2 The L^1 framework

Let Ω be a bounded domain with C^2 boundary and L the operator given by (2.1).

Definition 2.2 *We say that the operator L given by (2.1) satisfies the condition (H), if the functions a_{ij}, b_i and c_i are Lipschitz continuous in Ω, d is bounded and measurable, and if the uniform ellipticity condition (2.2) and the uniqueness condition (2.5) hold.*

Notice that this condition is symmetric in L and L^*. We put

$$\rho_{\partial\Omega}(x) = \mathrm{dist}\,(x, \partial\Omega), \quad \forall x \in \overline{\Omega}. \tag{2.17}$$

We denote by $C_c^{1,L}(\overline{\Omega})$ the space of $C^1(\overline{\Omega})$ functions ζ, vanishing on $\partial\Omega$ and such that $L^*\zeta \in L^\infty(\Omega)$, and by

$$\frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial \zeta}{\partial x_i}\mathbf{n}_j, \tag{2.18}$$

the co-normal derivative on the boundary following L^* (here the $\mathbf{n}_j$ are the components of outward normal unit vector $\mathbf{n}$ to $\partial\Omega$).

Definition 2.3 *Let $f \in L^1(\Omega; \rho_{\partial\Omega}dx)$ and $g \in L^1(\partial\Omega)$. We say that a function $u \in L^1(\Omega)$ is a very weak solution of the problem*

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{2.19}$$

if, for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, there holds

$$\int_\Omega uL^*\zeta dx = \int_\Omega f\zeta dx - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} g dS. \tag{2.20}$$

The next result is an adaptation of a construction, essentially due to Brezis in the case of the Laplacian, although various forms of existence theorems were known for a long time.

Theorem 2.4 *Let L satisfy the condition (H). Then for any f and g as in Definition 2.3, there exists one and only one very weak solution u of Problem (2.19). Furthermore, for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \geq 0$, there holds*

$$\int_\Omega |u| \, L^* \zeta dx \leq \int_\Omega f \mathrm{sign}(u) \zeta dx - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} |g| \, dS. \tag{2.21}$$

and

$$\int_\Omega u_+ L^* \zeta dx \leq \int_\Omega f \mathrm{sign}_+(u) \zeta dx - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} g_+ dS. \tag{2.22}$$

The following result shows the continuity of the process.

Lemma 2.5 *There exists a positive constant $C = C(L, \Omega)$ such that if f and g are as in Definition 2.3 and u is a very weak solution of (2.19),*

$$\|u\|_{L^1(\Omega)} \leq C \left(\|\rho_{\partial\Omega} f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)} \right). \tag{2.23}$$

Proof. We denote by η_u the solution of

$$\begin{aligned} L^* \eta_u &= \mathrm{sign}(u) \quad && \text{in } \Omega, \\ \eta_u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.24}$$

Notice that η_u exists by Lemma 2.1. Since the coefficients of L are Lipschitz continuous, $\eta_u \in C_c^1(\overline{\Omega})$ and $L^* \eta_u \in L^\infty(\Omega)$. Thus $\eta_u \in C_c^{1,L}(\overline{\Omega})$. By the maximum principle

$$|\eta_u| \leq \eta := \eta_1,$$

thus

$$\left| \frac{\partial \eta_u}{\partial \mathbf{n}_{L^*}} \right| \leq - \frac{\partial \eta}{\partial \mathbf{n}_{L^*}}.$$

Pluging this estimates into (2.20) one obtains

$$\int_\Omega |u| \, dx \leq \int_\Omega |f| \, \eta dx - \int_{\partial\Omega} \frac{\partial \eta}{\partial \mathbf{n}_{L^*}} |g| \, dS, \tag{2.25}$$

from which (2.23) follows. □

Proof of Theorem 2.4 -Existence Let $\{f_n\}$, $\{g_n\}$ be two sequences of C^2 functions defined respectively in Ω and $\partial\Omega$, f_n with compact support, and such that

$$\|(f - f_n)\rho_{\partial\Omega}\|_{L^1(\Omega)} + \|g - g_n\|_{L^1(\partial\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Let u_n be the classical solution (derived from Lemma 2.1 for example) of

$$\begin{aligned} Lu_n &= f_n \quad \text{in } \Omega, \\ u_n &= g_n \quad \text{on } \partial\Omega. \end{aligned} \tag{2.26}$$

Then $u_n \in W^{2,p}(\Omega)$ for any finite $p \geq 1$. By (2.23), $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega)$. Because u_n satisfies

$$\int_\Omega u_n L^* \zeta dx = \int_\Omega f_n \zeta dx - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} g_n dS, \tag{2.27}$$

for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, letting $n \to \infty$ leads to (2.20).

Estimates (2.21) and (2.22). Let γ be a smooth, odd and increasing function defined on $\mathbb{R}$ such that $-1 \leq \gamma \leq 1$, and ζ a nonnegative element of $C_c^{1,L}(\overline{\Omega})$. Since

$$\begin{aligned} \int_\Omega f_n \gamma(u_n) \zeta dx &= \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial(\gamma(u_n)\zeta)}{\partial x_i} dx \\ &+ \sum_i^n \int_\Omega \left(b_i \frac{\partial u_n}{\partial x_i} \gamma(u_n) \zeta + c_i u_n \frac{\partial(\gamma(u_n)\zeta)}{\partial x_i} \right) dx + \int_\Omega d u_n \gamma(u_n) \zeta dx \\ &\geq \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial \zeta}{\partial x_i} \gamma(u_n) \zeta dx \\ &+ \sum_i^n \int_\Omega \left(b_i \frac{\partial u_n}{\partial x_i} \gamma(u_n) \zeta + c_i u_n \frac{\partial(\gamma(u_n)\zeta)}{\partial x_i} \right) dx + \int_\Omega d u_n \gamma(u_n) \zeta dx. \end{aligned}$$

Put

$$j_1(r) = \int_0^r \gamma(s) ds, \; j_2(r) = r\gamma(r) \text{ and } j_3(r) = \int_0^r s\gamma'(s) ds.$$

Then

$$\begin{aligned} \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial \zeta}{\partial x_i} \gamma(u_n) dx &= \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial j_1(u_n)}{\partial x_j} \frac{\partial \zeta}{\partial x_i} dx \\ &= -\sum_{i,j=1}^n \int_\Omega j_1(u_n) \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial \zeta}{\partial x_i} \right) dx + \int_{\partial\Omega} j_1(g_n) \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} dS, \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^n \int_\Omega \left(b_i \frac{\partial u_n}{\partial x_i} \gamma(u_n) \zeta + c_i u_n \frac{\partial(\gamma(u_n)\zeta)}{\partial x_i} \right) dx \\ &\qquad = \sum_{i=1}^n \int_\Omega \left(b_i \frac{\partial j_1(u_n)}{\partial x_i} \zeta + c_i \left(j_2(u_n) \frac{\partial \zeta}{\partial x_i} + \zeta \frac{\partial j_3(u_n)}{\partial x_i} \right) \right) dx \\ &\qquad = \sum_{i=1}^n \int_\Omega \left(-j_1(u_n) \frac{\partial}{\partial x_i}(b_i \zeta) + c_i j_2(u_n) \frac{\partial \zeta}{\partial x_i} - j_3(u_n) \frac{\partial}{\partial x_i}(c_i \zeta) \right) dx. \end{aligned}$$

Therefore

$$\int_\Omega f_n\gamma(u_n)\zeta dx - \int_{\partial\Omega} j_1(g_n)\frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}dS \geq -\int_\Omega \left(\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial\zeta}{\partial x_i}\right) + \sum_{i=1}^n \frac{\partial}{\partial x_i}(b_i\zeta)\right) j_1(u_n)dx$$
$$+\int_\Omega \left(\sum_{i=1}^n c_i j_2(u_n)\frac{\partial\zeta}{\partial x_i} - j_3(u_n)\frac{\partial}{\partial x_i}(c_i\zeta) + dj_2(u_n)\zeta\right) dx,$$

and finally,

$$\int_\Omega j(u_n)L^*\zeta dx \leq \int_\Omega f_n\gamma(u_n)\zeta dx - \int_{\partial\Omega} j(g_n)\frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}dS.$$

When $\gamma(r) \to \mathrm{sign}(r)$, $j_1(r)$ and $j_2(r)$ both converge to $|r|$, and $j_3(r)$ converges to 0 if, for example, we impose $0 \leq \gamma'_\epsilon(r) \leq 2\epsilon^{-1}\chi_{(-\epsilon,\epsilon)}(r)$ and send ϵ to 0. Letting successively $n \to \infty$ and $\gamma \to$ sign yields to (2.21). We obtain (2.22) in the same way while approximating sign_+ by γ. $\square$

Corollary 2.6 *Under the assumptions of Theorem 2.4, the mapping $(f, g) \mapsto u$ defined by (2.19) is increasing.*

For the regularity of solutions, the following result is due to Brezis and Strauss [22] using Stampacchia's duality method [91].

Theorem 2.7 *Let L satisfy the condition (H). Then for any $1 \leq q < n/(n-1)$, there exists $C = C(\Omega, q) > 0$ such that for any $f \in L^1(\Omega)$, the very weak solution u of (2.19) with $g = 0$ satisfies*

$$\|u\|_{W_0^{1,q}(\Omega)} \leq C\,\|f\|_{L^1(\Omega)}. \tag{2.28}$$

This theorem admits a local version.

Corollary 2.8 *Let L be the elliptic operator defined by (2.1), with Lipschitz continuous coefficients and satisfying (2.2). Let $u \in L^1_{loc}(\Omega)$ and $f \in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega uL^*\zeta dx = \int_\Omega f\zeta dx, \tag{2.29}$$

for any $\zeta \in C^1_c(\Omega)$ such that $L^*\zeta \in L^\infty(\Omega)$. Then for any open subsets $G \subset \overline{G} \subset G' \subset \overline{G}' \subset \Omega$, with $\overline{G}'$ compact and $1 \leq q < n/(n-1)$, there exists a constant $C = C(G, G', q, L) > 0$ such that

$$\|u\|_{W^{1,q}(G)} \leq C\left(\|f\|_{L^1(G')} + \|u\|_{L^1(G')}\right). \tag{2.30}$$

2.3 The measure framework

We denote by $\mathfrak{M}(\Omega)$ and $\mathfrak{M}(\partial\Omega)$ the spaces of Radon measures on Ω and $\partial\Omega$ respectively, by $\mathfrak{M}_+(\Omega)$ and $\mathfrak{M}_+(\partial\Omega)$ their positive cones. For $0 \leq \alpha \leq 1$, we also denote by $\mathfrak{M}(\Omega; \rho_{\partial\Omega}^{\alpha})$ the subspace of the $\mu \in \mathfrak{M}(\Omega)$ satisfying

$$\int_\Omega \rho_{\partial\Omega}^{\alpha} d\,|\mu| < \infty,$$

and by $C(\overline{\Omega}; \rho_{\partial\Omega}^{-\alpha})$ the subspace of $C(\overline{\Omega})$ of functions ζ such that

$$\sup_\Omega |\zeta| / \rho_{\partial\Omega}^{\alpha} < \infty.$$

For the sake of clarity, we denote by

$$\mathfrak{M}(\Omega; \rho_{\partial\Omega}^{0}) = \mathfrak{M}^b(\Omega),$$

the space of bounded Radon measures in Ω. Both $\mathfrak{M}(\Omega; \rho_{\partial\Omega}^{\alpha})$ and $C(\overline{\Omega}; \rho_{\partial\Omega}^{-\alpha})$ are endowed with the norm corresponding to their definition. If $\lambda \in \mathfrak{M}(\Omega; \rho_{\partial\Omega})$ and $\mu \in \mathfrak{M}(\partial\Omega)$, the definition of a very weak solution to the measure data problem

$$\begin{aligned} Lu &= \lambda \quad \text{in } \Omega, \\ u &= \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{2.31}$$

is similar to Definition 2.3 : $u \in L^1(\Omega)$ and the equality

$$\int_\Omega u L^* \zeta dx = \int_\Omega \zeta d\lambda - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} d\mu, \tag{2.32}$$

holds for every $\zeta \in C_c^{1,L}(\overline{\Omega})$.

Theorem 2.9 *Let L satisfy the condition (H). For every $\lambda \in \mathfrak{M}(\Omega; \rho_{\partial\Omega})$ and $\mu \in \mathfrak{M}(\partial\Omega)$ there exists a unique very weak solution u to Problem (2.32). Furthermore the mapping $(\lambda, \mu) \mapsto u$ is increasing.*

Proof. Uniqueness follows from Lemma 2.5. For existence, let $\{\lambda_n\}$ be a sequence of smooth functions in $\overline{\Omega}$ such that

$$\lim_{n\to\infty} \int_\Omega \lambda_n \phi dx = \int_\Omega \phi d\lambda,$$

for every $\phi \in C(\overline{\Omega}; \rho_{\partial\Omega}^{-1})$. Let $\{\mu_n\}$ be a sequence of C^2 functions on $\partial\Omega$ converging to μ in the weak sense of measures and u_n denote the classical solution of

$$\begin{aligned} Lu_n &= \lambda_n \quad \text{in } \Omega, \\ u_n &= \mu_n \quad \text{on } \partial\Omega. \end{aligned} \tag{2.33}$$

Thus

$$\int_{\Omega} u_n L^*\zeta dx = \int_{\Omega} \zeta\lambda_n dx - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}\mu_n dS, \tag{2.34}$$

holds for every $\zeta \in C_c^{1,L}(\overline{\Omega})$. Since $\|\lambda_n\rho_{\partial\Omega}\|_{L^1(\Omega)}$ and $\|\mu_n\|_{L^1(\partial\Omega)}$ are bounded independently of n, it is the same with $\|u_n\|_{L^1(\Omega)}$ by Lemma 2.5. Let ω be a Borel subset of $\overline{\Omega}$, and $\theta_{\omega,n}$ the solution of

$$\begin{aligned} L^*\theta_{\omega,n} &= \chi_{\omega}\mathrm{sign}(u_n) && \text{in } \Omega, \\ \theta_{\omega,n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.35}$$

Since θ_ω is an admissible test function,

$$\int_{\omega} |u_n|\, dx = \int_{\Omega} \theta_{\omega,n}\lambda_n dx - \int_{\partial\Omega} \frac{\partial\theta_{\omega,n}}{\partial\mathbf{n}_{L^*}}\mu_n dS.$$

Moreover $-\theta_\omega \le \theta_{\omega,n} \le \theta_\omega$, where θ_ω is the solution of

$$\begin{aligned} L^*\theta_{\omega} &= \chi_{\omega} && \text{in } \Omega, \\ \theta_{\omega} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.36}$$

Therefore

$$\int_{\omega} |u_n|\, dx \le \|\lambda_n\rho_{\partial\Omega}\|_{L^1(\Omega)} \|\theta_\omega/\rho_{\partial\Omega}\|_{L^\infty(\Omega)} + \|\mu_n\|_{L^1(\partial\Omega)} \|\partial\theta_\omega/\partial\mathbf{n}_{L^*}\|_{L^\infty(\partial\Omega)}. \tag{2.37}$$

By the L^p regularity theory for elliptic equations and the Sobolev-Morrey imbedding Theorem, for any $n < p < \infty$, there exists a constant $C = C(n,p) > 0$ such that

$$\|\theta_\omega\|_{C^1(\overline{\Omega})} \le C\|\chi_\omega\|_{L^p(\Omega)} = C|\omega|^{1/p}. \tag{2.38}$$

This estimate, combined with (2.37), yields to

$$\int_{\omega} |u_n|\, dx \le C(\|\lambda_n\rho_{\partial\Omega}\|_{L^1(\Omega)} + \|\mu_n\|_{L^1(\partial\Omega)})|\omega|^{1/p} \le CM|\omega|^{1/p}, \tag{2.39}$$

for some M independent of n. Therefore the sequence $\{u_n\}$ is uniformly integrable, thus weakly compact in $L^1(\Omega)$ by the Dunford-Pettis Theorem, and there exist a subsequence $\{u_{n_k}\}$ and an integrable function u such that $u_{n_k} \to u$, weakly in $L^1(\Omega)$. Passing to the limit in (2.34) leads to (2.32). Because of uniqueness the whole sequence $\{u_n\}$ converges weakly to u. The monotonicity assertion follows from uniqueness and Corollary 2.6. □

Remark. Estimate (2.22) in the statement of Theorem 2.4 admits the following extension : Let the two measures λ and μ have Lebesgue decomposition

$$\lambda = \lambda_r + \lambda_s \quad \text{and } \mu = \mu_r + \mu_s,$$

λ_r and μ_r being the regular parts with respect to the n and the $n-1$ dimensional Hausdorff measures and λ_s and μ_s the singular parts. If λ_s and μ_s are nonpositive, there holds

$$\int_{\Omega} u_+ L^*\zeta dx \le \int_{\Omega} \lambda_{r+} sign_+(u)\zeta dx - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}\mu_{r+} dS, \tag{2.40}$$

for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \geq 0$.

Remark. The above proof implies the following weak stability result. *If $\{\lambda_n\} \subset \mathfrak{M}(\Omega; \rho_{\partial\Omega})$ and $\{\mu_n\} \subset \mathfrak{M}(\partial\Omega)$ are sequences of measures wich converge respectively to λ in duality with $C(\overline{\Omega}; \rho_{\partial\Omega}^{-1})$, and to μ in the weak sense of measures on $\partial\Omega$, the corresponding very weak solutions u_n of (2.33) converge weakly in $L^1(\Omega)$ to the very weak solution u of (2.31).*

2.4 Representation theorems and boundary trace

If Ω is a bounded domain with a C^2 boundary, L the elliptic operator defined by (2.1), with Lispchitz countinuous coefficients and u and v two functions in $W^{2,p}(\Omega)$, with $p > n$, the Green formula implies

$$\int_\Omega (vLu - uL^*v)\,dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \mathbf{n}_{L^*}} - v\frac{\partial u}{\partial \mathbf{n}_L} \right) dS, \tag{2.41}$$

where L^* and $\partial v/\partial \mathbf{n}_{L^*}$ are respectively defined by (2.15) and (2.18), and

$$\frac{\partial \zeta}{\partial \mathbf{n}_L} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial \zeta}{\partial x_j}\mathbf{n}_i, \tag{2.42}$$

is the co-normal derivative following L. If we assume that condition (H) is fulfilled, and if $x \in \Omega$, we denote by $G_L^\Omega(x,.)$ the solution of

$$\begin{aligned} L^*G_L^\Omega(x,.) &= \delta_x \qquad \text{in } \Omega,\\ G_L^\Omega(x,.) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.43}$$

The function G_L^Ω is the *Green function* of the operator L in Ω. Notice an ambiguity in terminology between L and L^*, but it has no consequence because the condition (H) is invariant by duality and the following symmetry result holds :

$$G_L^\Omega(x,y) = G_{L^*}^\Omega(y,x), \quad \forall (x,y) \in \Omega \times \Omega,\ x \neq y. \tag{2.44}$$

The function $G_L^\Omega(x,.)$ is nonnegative by Theorem 2.9 and belongs to $W_{loc}^{2,p}(\Omega \setminus \{x\})$ for any $1 < p < \infty$. Thus it is C^1 in $\overline{\Omega} \setminus \{x\}$. We denote

$$P_L^\Omega(x,y) = -\frac{\partial G_L^\Omega(x,y)}{\partial \mathbf{n}_{L^*}}, \quad \forall (x,y) \in \Omega \times \partial\Omega. \tag{2.45}$$

If $u \in C^2(\bar{\Omega})$, the following Green representation formula derives from (2.41)

$$u(x) = \int_\Omega G_L^\Omega(x,y)Lu(y)dy + \int_{\partial\Omega} P_L^\Omega(x,y)u(y)dS(y), \quad \forall x \in \Omega. \tag{2.46}$$

By extension this representation formula holds almost everywhere if $(\lambda,\mu) \in \mathfrak{M}(\Omega) \times (\partial\Omega)$, and u is the very weak solution of (2.31), in the sense that

$$u(x) = \int_\Omega G_L^\Omega(x,y)d\lambda(y) + \int_\Omega G_L^\Omega(x,y)d\mu(y), \quad \text{a.e. in } \Omega. \tag{2.47}$$

Actually the representation formula is equivalent to the fact that u is a very weak solution of Problem 2.31 (see [14] for a proof). We set

$$\mathbb{G}_L^\Omega(\lambda)(x) = \int_\Omega G_L^\Omega(x,y)d\lambda(y), \tag{2.48}$$

and call it the *Green potential* of λ, and

$$\mathbb{P}_L^\Omega(\lambda)(x) = \int_{\partial\Omega} P_L^\Omega(x,y)d\lambda(y), \quad \forall x \in \Omega, \tag{2.49}$$

the *Poisson potential* of μ. The Green kernel presents a singularity on the diagonal $D_\Omega = \{(x,x) : x \in \Omega\}$, while the Poisson kernel becomes singular when the x variable approaches the boundary point y. Many estimates on the singularities have been obtained in the last thirty years [56], [78], [47], [35]. We give below some useful estimates in which $\rho_{\partial\Omega}$ is defined by (2.17).

Theorem 2.10 *Assuming that Ω is bounded with a C^2 boundary and assumption (H) holds, then*

$$G_L^\Omega(x,y) \leq C(L,\Omega)\frac{\min\{1, |x-y|\,\rho_{\partial\Omega}(x)\}}{|x-y|^{n-2}}, \quad \forall (x,y) \in (\Omega\times\Omega)\setminus D_\Omega, \tag{2.50}$$

if $n \geq 3$,

$$G_L^\Omega(x,y) \leq C(L,\Omega)\min\{1, |x-y|\,\rho_{\partial\Omega}(x)\}\ln_+|x-y|, \quad \forall (x,y) \in (\Omega\times\Omega)\setminus D_\Omega, \tag{2.51}$$

if $n = 2$. Moreover, for any $n \geq 2$,

$$K'(L,\Omega)\frac{\rho_{\partial\Omega}(x)}{|x-y|^n} \leq P_L^\Omega(x,y) \leq K(L,\Omega)\frac{\rho_{\partial\Omega}(x)}{|x-y|^n}, \quad \forall (x,y) \in \Omega\times\partial\Omega. \tag{2.52}$$

Another useful notion, from which some of the above estimates can be derived is the notion of equivalence (see [4], [85]).

Theorem 2.11 *Assuming that Ω is bounded with a C^2 boundary and assumption (H) holds, there exists a positive constant C such that*

$$CG_{-\Delta}^\Omega \leq G_L^\Omega \leq \frac{1}{C}G_{-\Delta}^\Omega \quad \text{in } (\Omega\times\Omega)\setminus D_\Omega, \tag{2.53}$$

and

$$CP_{-\Delta}^\Omega \leq P_L^\Omega \leq \frac{1}{C}P_{-\Delta}^\Omega \quad \text{in } \Omega\times\partial\Omega. \tag{2.54}$$

In order to study the boundary behaviour of harmonic functions, we introduce, for $\beta > 0$,

$$\Omega_\beta = \{x \in \Omega : \rho_\Omega(x) > \beta\}, \quad G_\beta = \Omega\setminus\overline{\Omega}_\beta, \quad \Sigma_\beta = \partial\Omega_\beta = \{x \in \Omega : \rho_\Omega(x) = \beta\}, \tag{2.55}$$

and $\Sigma_0 := \Sigma := \partial\Omega$. Since Ω is C^2, there exists $\beta_0 > 0$ such that for every $0 < \beta \leq \beta_0$ and $x \in G_\beta$ there exists a unique $\sigma(x) \in \Sigma$ such that $|x - \sigma(x)| = \rho_{\partial\Omega}(x)$. We denote by Π the mapping from G_β to $(0, \beta) \times \Sigma$ defined by

$$\Pi(x) = (\rho_{\partial\Omega}(x), \sigma(x)). \tag{2.56}$$

The mapping Π is a C^2 diffeomorphism with inverse given by

$$\Pi^{-1}(t, \sigma) = \sigma - t\mathbf{n}, \quad \forall (t, \sigma) \in (0, \beta) \times \Sigma, \tag{2.57}$$

where $\mathbf{n}$ is the normal unit outward vector to $\partial\Omega$ at x (see [71] for details). If the distance coordinate is fixed in $(0, \beta_0]$, the mapping $\mathfrak{H}_t$ defined by

$$\mathfrak{H}_t(x) = \sigma(x) \quad \forall x \in \Sigma_t,$$

is the orthogonal projection from Σ_t to $\partial\Omega$. Thus $\mathfrak{H}_t^{-1}(.) = \Pi^{-1}(t, .)$ is a C^2 diffeomorphism and the set $\{\Sigma_t\}_{0<t\leq\beta_0}$ is a C^2 foliation of G_{β_0}. For $0 < t \leq \beta_0$ we can transfer naturally a Borel measure μ, or a function f, on Σ_t into a Borel measure or a function on $\partial\Omega$ as follows :

$$\begin{aligned} &\mu^t(E) := \mu(\mathfrak{H}_t^{-1}(E)), \quad \text{for every Borel subset } \mathrm{E} \subset \partial\Omega, \\ &f^t(x) := f(\sigma(x) - t\mathbf{n}(x)), \quad \forall x \in \partial\Omega. \end{aligned} \tag{2.58}$$

The Lebesgue classes on Σ_t and Σ are exchanged by this projection operator and actually

$$\mu \in \mathfrak{M}(\Sigma_t),\ f \in L^1(\Sigma_t, \mu) \Longrightarrow \left\{ \begin{array}{l} f^t \in L^1(\Sigma, \mu^t), \\ \displaystyle\int_{\Sigma_t} f d\mu = \int_\Sigma f^t d\mu^t. \end{array} \right. \tag{2.59}$$

Definition 2.12 Let L be an elliptic operator defined by (2.1) in Ω, with bounded and measurable coefficients. We say that a function $u \in W^{1,2}_{loc}(\Omega)$ is weakly L-harmonic if

$$A_L(u, v) = 0 \quad \forall v \in C^1_c(\Omega). \tag{2.60}$$

Remark. If (2.2) holds, any weakly L-harmonic function is Hölder continuous by the De Giorgi-Nash-Moser Theorem. If the coefficients of L are Lispchitz continuous, the notion of weak L-harmonicity can be understood in the sense of distributions in Ω, by assuming that u is locally integrable in Ω and

$$\int_\Omega u L^* \phi dx = 0, \quad \forall \phi \in C^\infty_0(\Omega). \tag{2.61}$$

It can be verified that any locally integrable function L-harmonic in the sense of distributions in Ω is actually weakly L-harmonic. Therefore it belongs to $W^{2,p}_{loc}(\Omega)$, for any $1 < p < \infty$, by the L^p-regularity theory of elliptic equations.

Theorem 2.13 *Let Ω be a bounded domain of class C^2 and L the elliptic operator defined by (2.1) satisfying condition (H). Let u be a nonnegative locally integrable L-harmonic function in Ω. Then there exists a unique nonnegative Radon measure μ on $\partial\Omega$ such that*

$$\lim_{t\to 0} \int_{\Sigma_t} u(x)\theta(\sigma(x))dS = \int_\Sigma \theta d\mu, \quad \forall \theta \in C_0(\Sigma). \tag{2.62}$$

Moreover u is uniquely determined by μ and

$$u(x) = \int_{\partial\Omega} P_L^{\Omega}(x,y)d\mu(y), \quad \forall x \in \Omega. \tag{2.63}$$

Proof. Step 1 The function u is integrable. Let $0 < \beta \leq \beta_0$. Since u is continuous in $\overline{\Omega}_\beta$, its restriction to this set is the very weak solution of

$$\begin{aligned} Lv &= 0 \qquad \text{in } \Omega_\beta, \\ v &= u\lfloor_{\Sigma_\beta} \quad \text{on } \Sigma_\beta. \end{aligned} \tag{2.64}$$

Thus, if $\zeta \in C_c^{1,L}(\overline{\Omega}_\beta)$, there holds

$$\int_{\Omega_\beta} uL^*\zeta dx = -\int_{\Sigma_\beta} \frac{\partial\zeta}{\partial \mathbf{n}_{L^*}} udS. \tag{2.65}$$

We fix $\zeta = \eta_{1,\beta}$ as the solution of

$$\begin{aligned} L^*\eta_{1,\beta} &= 1 \quad \text{in } \Omega_\beta, \\ \eta_{1,\beta} &= 0 \quad \text{on } \Sigma_\beta. \end{aligned} \tag{2.66}$$

By Hopf's lemma, there exists $c > 0$ such that

$$c \leq -\frac{\partial\eta_{1,\beta}}{\partial\mathbf{n}_{L^*}} \leq c^{-1} \quad \text{on } \Sigma_\beta.$$

Moreover, c can be taken independent of $\beta \in (0, \beta_0]$. It follows

$$\Psi(\beta) = \int_{\Omega_\beta} udx \geq c\int_{\Sigma_\beta} udS = -c\Psi'(\beta), \tag{2.67}$$

from (2.65), and

$$\frac{d}{d\beta}\left(e^{\beta/c}\Psi(\beta)\right) \geq 0.$$

Therefore

$$\lim_{\beta\to 0} \Psi(\beta) = \int_\Omega udx < \infty.$$

Notice that (2.67) implies that $\|u\|_{L^1(\Sigma_\beta)}$ remains bounded independently of β.

Step 2 End of the proof. Let $\theta \in C^2(\partial\Omega)$, w_θ be the solution of

$$\begin{aligned} L^*w_\theta &= 0 \quad \text{in } \Omega_\beta, \\ w_\theta &= \theta \quad \text{on } \Sigma_\beta, \end{aligned} \tag{2.68}$$

and $h \in C(\Sigma_\beta)$ defined by

$$h = -\frac{\partial\eta_{1,\beta}}{\partial\mathbf{n}_{L^*}}.$$

Then $\zeta = \eta_{1,\beta} w_\theta h^{-1}$ belongs to $C_c^{1,L}(\overline{\Omega}_\beta)$. Since $\partial\zeta/\partial\mathbf{n}_{L^*} = \theta$ on Σ_β,

$$\int_{\Omega_\beta} uL^*\zeta dx = -\int_{\Sigma_\beta} \frac{\partial\zeta}{\partial\mathbf{n}_{L^*}} dS = \int_{\Sigma_\beta} \theta u dS.$$

It is easy to check that $L^*\zeta$ is bounded in $L^\infty(\Omega_\beta)$, independently of β. Therefore

$$\lim_{\beta\to 0} \int_{\Omega_\beta} uL^*\zeta dx$$

exists. The same holds true with

$$\lim_{\beta\to 0} \int_{\Sigma_\beta} \theta u dS,$$

which defines a positive linear functional on $C^2(\partial\Omega)$. This characterizes the Radon measure μ in a unique way. □

Definition 2.14 The measure μ is called *the boundary trace* of u.

Remark. In the above theorem, many assumptions can be relaxed : the boundedness of Ω plays no role except that it allows a simpler statement of the result, and the integral represention (2.63). The regularity of the boundary of the domain is not a key assumption, but in the case of a general domain, the boundary has to be replaced by the Martin boundary [76], and the Poisson kernel by the Martin kernel in order to have a representation formula valid for all the positive L-harmonic functions.

Remark. The Fatou Theorem asserts that for almost all $y \in \partial\Omega$ (for the $n-1$-dimensional Hausdorff measure dH_{n-1}) and for any cone C_y interior to Ω the following limit exists,

$$\lim_{\substack{x \to y \\ x \in C_y}} u(x) = \mu_r, \tag{2.69}$$

where μ_r is the regular part of the measure μ with respect to dH_{n-1} in the Lebesgue decomposition. The proof of this result [30], [32], is much more involved that the one of Theorem 2.13. The trace in the sense of Radon measures is much more useful for our next considerations.

Definition 2.15 A locally integrable function u defined in Ω is said super-L-harmonic if

$$\int_\Omega uL^*\phi dx \geq 0, \quad \forall\phi \in C_0^\infty(\Omega),\ \phi \geq 0. \tag{2.70}$$

Theorem 2.13 admits an extension to positive proven by Doob [60], [32].

Theorem 2.16 *Let Ω be a bounded domain of class C^2 and L the elliptic operator defined by (2.1). We assume that condition (H) holds. Let u a nonnegative super-L-harmonic in Ω. Then there exist two Radon measures $\lambda \in \mathfrak{M}_+(\Omega)$ and $\mu \in \mathfrak{M}_+(\partial\Omega)$, such that*

$$\int_\Omega \rho_{\partial\Omega} d\lambda < \infty,$$

and u is the unique very weak solution to Problem (2.31). Furthermore (2.69) holds.

3 Semilinear equations with absorption

In this section we consider the semilinear Dirichlet problem with right-hand side measure

$$\begin{aligned} Lu + g(x,u) &= \lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

in a bounded domain Ω of $\mathbb{R}^n$, where g is a continuous function defined on $\mathbb{R} \times \Omega$, λ a Radon measure in Ω and L the elliptic operator with Lipschitz continuous coefficients, defined by (2.1).

Definition 3.1 *Let $\lambda \in \mathfrak{M}(\Omega;\rho_{\partial\Omega})$. A function u is a solution of (3.1), if $u \in L^1(\Omega)$, $g(.,u) \in L^1(\Omega;\rho_{\partial\Omega}dx)$, and if for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, there holds*

$$\int_\Omega (uL^*\zeta + g(x,u)\zeta)\,dx = \int_\Omega \zeta d\lambda. \tag{3.2}$$

The nonlinearity is understood as an absorption term, this means that $rg(x,r)$ is nonnegative for $|r| \geq r_0$, uniformly with respect to $x \in \Omega$.

Proposition 3.2 *Let L be the elliptic operator defined by (2.1), satisfying the condition (H), and $\lambda \in \mathfrak{M}(\Omega;\rho_{\partial\Omega}^\alpha)$ for some $0 \leq \alpha \leq 1$. If $g \in C(\Omega,\mathbb{R})$ is an absorption nonlinearity satisfying*

$$rg(x,r) \geq 0, \quad \forall (x,r) \in \Omega \times ((-\infty,-r_0] \cup [r_0,\infty)),$$

and g bounded on $\Omega \times (-r_0,r_0)$, any solution u of (3.1) verifies $g(.,u) \in L^1(\Omega;\rho_{\partial\Omega}^\alpha dx)$.

Proof. We set $h = g(.,u)$, then u is the unique very weak solution of

$$Lu = \lambda - h \quad \text{in } \Omega,$$

and, by assumption, $u \in L^1(\Omega)$, $h \in L^1(\Omega;\rho_{\partial\Omega}dx)$. Let $\{\lambda_n\}$ be a sequence of smooth functions in $\overline{\Omega}$ converging to λ in the weak sense of measures with duality with the space $C(\overline{\Omega};\rho_{\partial\Omega}^{-\alpha})$ (thus $\|\lambda_n\|_{\mathfrak{M}(\Omega;\rho_{\partial\Omega}^\alpha)}$ is bounded independently of n) and $\{u_n\}$ the corresponding sequence of solutions of

$$Lu_n = \lambda_n - h \quad \text{in } \Omega.$$

By Theorem 2.4, $\|u_n\|_{L^1(\Omega)}$ is bounded independently of n, and for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \geq 0$, there holds

$$\int_\Omega (|u_n|\, L^*\zeta + h\zeta\operatorname{sign}(u_n))\,dx \leq \int_\Omega \lambda_n\zeta\operatorname{sign}(u_n)dx.$$

For test function ζ, we take $j_\epsilon(\eta_1) = (\eta_1+\epsilon)^\alpha - \epsilon^\alpha$ where $\epsilon > 0$. Then $0 \leq j_\epsilon(\eta_1) \leq \eta_1^\alpha$, and, if we put $r_1 = \sup_\Omega \eta_1$, one obtains

$$
\begin{aligned}
L^*(j_\epsilon(\eta_1)) &= -j'_\epsilon(\eta_1)\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \eta_1}{\partial x_i}\right) + j'_\epsilon(\eta_1)\sum_{i=1}^n c_i \frac{\partial \eta_1}{\partial x_i} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(b_i j_\epsilon(\eta_1)\right) + d j_\epsilon(\eta_1) \\
&\quad - j''_\epsilon(\eta_1)\sum_{i,j=1}^n a_{ij}\frac{\partial \eta_1}{\partial x_j}\frac{\partial \eta_1}{\partial x_i} \\
&= j'_\epsilon(\eta_1)L^*\eta_1 + \left(j_\epsilon(\eta_1) - \eta_1 j'_\epsilon(\eta_1)\right)\left(d - \sum_{i=1}^n \frac{\partial b_i}{\partial x_i}\right) - j''_\epsilon(\eta_1)\sum_{i,j=1}^n a_{ij}\frac{\partial \eta_1}{\partial x_j}\frac{\partial \eta_1}{\partial x_i} \\
&\geq -\left(j_\epsilon(r_1) - r_1 j'_\epsilon(r_1)\right)\left|d - \sum_{i=1}^n \frac{\partial b_i}{\partial x_i}\right|,
\end{aligned}
$$

since $L^*\eta_1 = 1$, j_ϵ is a concave and increasing function on $\mathbb{R}_+$, $r \mapsto j_\epsilon(r) - r j'_\epsilon(r)$ is positive and increasing, and ellipticity condition (2.2) holds. Because $(j_\epsilon(r_1) - r_1 j'_\epsilon(r_1))$ is bounded when $0 < \epsilon \leq 1$, and the coefficients b_i and d are respectively Lipschitz continuous and bounded in Ω, there exists a constant $M > 0$ independent of ϵ such that $L^*(j_\epsilon(\eta_1)) \geq -M$ in Ω. Therefore

$$
-M\int_\Omega |u_n|\, dx + \int_\Omega h j_\epsilon(\eta_1)\mathrm{sign}(u_n) dx \leq \|\lambda_n\|_{\mathfrak{M}(\Omega;\rho^\alpha_{\partial\Omega})}.
$$

Letting $n \to \infty$ yields to

$$
\int_\Omega g(x,u) j_\epsilon(\eta_1)\mathrm{sign}(u) dx \leq M\int_\Omega |u|\, dx + \sup_n \|\lambda_n\|_{\mathfrak{M}(\Omega;\rho^\alpha_{\partial\Omega})}, \tag{3.3}
$$

since $h \in L^1(\Omega; \rho_\Omega dx)$. To be more precise, it is necessary to take a sequence of smooth approximations γ_κ of the function sign, then let $\kappa \to 0$ and $\gamma_\kappa \to$ sign as in the proof of Theorem 2.4. Therefore there exists a positive constant C such that

$$
\int_{\{x:|u(x)|\geq r_0\}} g(x,u) j_\epsilon(\eta_1)\mathrm{sign}(u) dx \leq C + \int_{\{x:|u(x)|<r_0\}} g(x,u) j_\epsilon(\eta_1)\mathrm{sign}(u) dx.
$$

Using the fact that $g(x,r)\mathrm{sign}(r)$ is positive for $|r| \geq r_0$ and bounded for $|r| < r_0$, we can let $\epsilon \to 0$ and conclude, thanks to Fatou's lemma, that

$$
\int_\Omega |g(x,u)|\, \eta_1^\alpha dx < C + \sup_n \|\lambda_n\|_{\mathfrak{M}(\Omega;\rho^\alpha_{\partial\Omega} dx)} < \infty, \tag{3.4}
$$

which ends the proof. □

3.1 The Marcinkiewicz spaces approach

At first we recall some definitions and basic properties of the Marcinkiewicz spaces. Let G be an open subset of $\mathbb{R}^d$ and λ a positive Borel measure on G.

Definition 3.3 For $p > 1$, $p' = p/(p-1)$ and $u \in L^1_{loc}(G)$, we introduce

$$\|u\|_{M^p(G;d\lambda)} = \inf\left\{c \in [0,\infty] : \int_E |u|\, d\lambda \leq c\left(\int_E d\lambda\right)^{1/p'}, \ \forall E \subset G,\, E \text{ Borel}\right\}, \quad (3.5)$$

and

$$M^p(G;d\lambda) = \{u \in L^1(G;d\lambda) : \|u\|_{M^p(G;d\lambda)} < \infty\}. \quad (3.6)$$

$M^p(G;d\lambda)$ is called the Marcinkiewicz space of exponent p, or weak L^p-space. It is a Banach space and the following estimates can be found in [12] and [26].

Proposition 3.4 *Let $1 \leq q < p < \infty$ and $u \in L^1_{loc}(G;d\lambda)$. Then*

$$C(p)\,\|u\|_{M^p(G;d\lambda)} \leq \sup\left\{s > 0 : s^p \int_{\{x:\,|u(x)|>s\}} d\lambda\right\} \leq \|u\|_{M^p(G;d\lambda)}\,. \quad (3.7)$$

Furthermore

$$\int_E |u|^q\, d\lambda \leq C(p,q)\,\|u\|_{M^p(G;d\lambda)}\left(\int_E d\lambda\right)^{1-q/p}, \quad (3.8)$$

for any Borel set $E \subset G$.

The key role of Marcinkiewicz spaces is to give optimal estimates when solving elliptic equations in a measure framework. In particular, using (2.50) and (2.52) it is not difficult to prove the following result (see [14] for a more general set of estimates in the case of the Laplacian operator).

Theorem 3.5 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a C^2 bounded domain and L an elliptic operator satisfying condition (H). Let $\alpha \in [0,1]$, $\lambda \in \mathfrak{M}(\Omega;\rho^\alpha_{\partial\Omega})$, $\mu \in \mathfrak{M}(\partial\Omega)$. If $n + \alpha > 2$, there holds*

$$\left\|\mathbb{G}^\Omega_L(\lambda)\right\|_{M^{(n+\alpha)/(n+\alpha-2)}(\Omega;\rho^\alpha_{\partial\Omega})} \leq C\|\lambda\|_{\mathfrak{M}(\Omega;\rho^\alpha_{\partial\Omega})}, \quad (3.9)$$

and

$$\left\|\nabla\mathbb{G}^\Omega_L(\lambda)\right\|_{M^{(n+\alpha)/(n+\alpha-1)}(\Omega;\rho^\alpha_{\partial\Omega})} \leq C\|\lambda\|_{\mathfrak{M}(\Omega;\rho^\alpha_{\partial\Omega})}. \quad (3.10)$$

Furthermore, for any $\gamma \in [0,1]$,

$$\left\|\mathbb{P}^\Omega_L(\mu)\right\|_{M^{(n+\gamma)/(n-1)}(\Omega;\rho^\gamma_{\partial\Omega})} \leq C\|\mu\|_{\mathfrak{M}(\partial\Omega)}. \quad (3.11)$$

The following definition is inspired by Bénilan and Brezis classical work [11] (with $\alpha = 0$) and used by Gmira and Véron [48] (with $\alpha = 1$).

Definition 3.6 A real valued function $g \in C(\Omega \times \mathbb{R})$ satisfies the (n, α)-*weak-singularity assumption*, $n \geq 2$, $\alpha \in [0, 1]$, $n + \alpha > 2$, if there exists $r_0 \geq 0$ such that

$$rg(x, r) \geq 0, \quad \forall (x, r) \in \Omega \times (-\infty, -r_0] \cup [r_0, \infty), \tag{3.12}$$

and a nondecreasing function $\tilde{g} \in C([0, \infty))$ such that $\tilde{g} \geq 0$,

$$\int_0^1 \tilde{g}(r^{2-n-\alpha}) r^{n+\alpha-1} dr < \infty, \tag{3.13}$$

and

$$|g(x, r)| \leq \tilde{g}(|r|), \quad \forall (x, r) \in \Omega \times \mathbb{R}. \tag{3.14}$$

Theorem 3.7 *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 2$, L the elliptic operator defined by (2.1) and $g \in C(\Omega \times \mathbb{R})$ a real valued function. If L satisfies assumptions (H) and g the (n, α)-weak-singularity assumption (then $n \geq 3$ if $\alpha = 0$), for any $\lambda \in \mathfrak{M}(\Omega; \rho_{\partial\Omega}^{\alpha})$ there exists a solution u to Problem (3.1).*

Proof. *Step 1* Construction of approximate solutions. The technique developed below is adapted from Brezis and Strauss classical article [22]. Let λ_n be a sequence of smooth functions, with compact support in Ω, with uniformly bounded $L^1(\Omega; \rho_{\partial\Omega} dx)$-norm, with the property

$$\lim_{n \to \infty} \int_{\Omega} \lambda_n \zeta dx \to \int_{\Omega} \zeta d\lambda,$$

for any $\zeta \in C(\overline{\Omega})$ such that $\sup_{\Omega}(\rho_{\partial\Omega}^{-\alpha} |\zeta|) < \infty$. For $k > 0$, we introduce the truncation $g_k(., r)$ of $g(., r)$ by

$$g_k(x, r) = \begin{cases} g(x, r) & \text{if } |g(x, r)| \leq k, \\ k \,\mathrm{sign}(g(x, r)) & \text{if } |g(x, r)| > k. \end{cases} \tag{3.15}$$

By Lax-Milgram's theorem, for any $z \in L^2(\Omega)$, there exists a unique $w = \mathcal{T}_k(z)$ such that

$$A_L(w, \phi) + \int_{\Omega} g_k(x, z) \phi dx = \int_{\Omega} \lambda_n \phi dx, \quad \forall \phi \in W_0^{1,2}(\Omega). \tag{3.16}$$

Using (2.2),

$$\alpha \|\nabla w\|_{L^2(\Omega)}^2 \leq \left(k |\Omega|^{1/2} + \|\lambda_n\|_{L^2(\Omega)} \right) \|w\|_{L^2(\Omega)}.$$

The mapping $\mathcal{T}_k$ is continuous in $L^2(\Omega)$. By the above estimate and Rellich-Kondrachov's theorem, $\mathcal{T}_k$ sends $L^2(\Omega)$ into a relatively compact subset of $L^2(\Omega)$. By Schauder's theorem, it admits a fixed point, say $v = v_k$, and v_k solves

$$L v_k + g_k(., v_k) = \lambda_n \quad \text{in } \Omega. \tag{3.17}$$

The functions v_k belongs to $C_c^{1,L^*}(\overline{\Omega})$, since λ_n and g_k are bounded. Multiplying by v_k and using (3.14) (one notices that the two inequalities are uniform with respect to k), yields to

$$\alpha \|\nabla v_k\|_{L^2(\Omega)}^2 \leq \left(\Theta |\Omega|^{1/2} + \|\lambda_n\|_{L^2(\Omega)} \right) \|v_k\|_{L^2(\Omega)},$$

since $rg(x,r) \geq -\Theta\,|r|$, for some Θ verifying

$$0 \leq \Theta \leq \sup\{|g(x,r)| : \, x \in \Omega, \, -r_0 \leq r \leq r_0\}. \tag{3.18}$$

Hence the set of functions $\{v_k\}$ remains bounded in $W_0^{1,2}(\Omega)$ independently of k.

Step 2 Uniform estimates. In order to prove that there exists some k such that v_k satisfies

$$Lv_k + g(.,v_k) = \lambda_n \quad \text{in } \Omega. \tag{3.19}$$

it is sufficient to prove that v_k is uniformly bounded in Ω. The technique used is due to Moser [79]. For $\theta \geq 1$, $|v_k|^{\theta-1} v_k$ belongs to $W_0^{1,2}(\Omega)$. For simplicity we denote it by v_k^θ, thus

$$A_L(v_k, v_k^\theta) + \int_\Omega g_k(x,v_k) v_k^\theta dx = \int_\Omega \lambda_n v_k^\theta dx. \tag{3.20}$$

But, using (2.2) and (2.5),

$$\begin{aligned} A_L(v_k, v_k^\theta) &\geq \alpha\theta \int_\Omega |\nabla v_k|^2 v_k^{\theta-1} dx + \sum_{i=1}^n \int_\Omega (b_i + \theta c_i) v_k^\theta \frac{\partial v_k}{\partial x_i} dx + \int_\Omega d v_k^{\theta+1} dx \\ &\geq \frac{4\alpha\theta}{(\theta+1)^2} \int_\Omega \left|\nabla\left(|v_k|^{(\theta+1)/2}\right)\right|^2 dx + \frac{\theta-1}{2} \sum_{i=1}^n \int_\Omega (c_i - b_i) \frac{\partial v_k}{\partial x_i} v_k^\theta dx \\ &\geq \frac{4\alpha\theta}{(\theta+1)^2} \int_\Omega \left|\nabla\left(|v_k|^{(\theta+1)/2}\right)\right|^2 dx - \frac{\theta-1}{2(\theta+1)} \int_\Omega |v_k|^{\theta+1} \mathrm{div}\mathcal{H}\mathrm{dx} \end{aligned}$$

where $\mathcal{H}_i = c_i - b_i$ and

$$\int_\Omega g_k(x,v_k) v_k^\theta dx \geq -\Theta \int_\Omega |v_k|^\theta dx.$$

By using the previous estimates and Gagliardo-Nirenberg's inequality, it follows that, for some $\sigma > 0$ and $C_i \geq 0$ depending on λ_n but not on k, there holds

$$\begin{aligned} \frac{\sigma\theta}{(\theta+1)^2} \|v_k\|^{\theta+1}_{L^{(\theta+1)n/(n-2)}(\Omega)} &\leq C_1 \|v_k\|^\theta_{L^{\theta+1}(\Omega)} + C_2 \|v_k\|^{\theta+1}_{L^{\theta+1}(\Omega)} \\ &\leq C_3 \max\{1, \|v_k\|^{\theta+1}_{L^{\theta+1}(\Omega)}\}. \end{aligned}$$

Putting $a = n/(n-2)$, $\gamma = \theta + 1$,

$$\|v_k\|_{L^{a\gamma}(\Omega)} \leq C_4^{1/\gamma} \gamma^{2/\gamma} \max\{1, \|v_k\|_{L^\gamma(\Omega)}\}.$$

Iterating from $\gamma = 2$, we obtain

$$\begin{aligned} \|v_k\|_{L^{a^{m+1}\gamma}(\Omega)} &\leq C_5^{\Sigma_{j=0}^m a^{-j}} 2^{\Sigma_{j=0}^m j a^{-j}} \max\{1, \|v_k\|_{L^2(\Omega)}\} \\ &\leq C_6 \max\{1, \|v_k\|_{L^2(\Omega)}\}. \end{aligned}$$

Consequently $|v_k(x)|$ is uniformly bounded by some k_0. Taking $k > k_0$, v_k is a solution of

$$Lv_k + g(.,v_k) = \lambda_n \quad \text{in } \Omega. \tag{3.21}$$

In order to emphasize the fact that v_k is actually independent of k, but not on n, we shall denote it by u_n.

Step 3 Uniform integrability. It follows from Step 2 that $g(.,u_n)u_n$ is integrable in Ω and the same is true with $g(.,u_n)$, because of (3.14). The space $C_c^{1,L}(\overline{\Omega})$ is a subspace of $W_0^{1,2}(\Omega)$, therefore (3.16) implies

$$\int_\Omega (u_n L^*\zeta + g(x,u_n)\zeta)\,dx = \int_\Omega \lambda_n \zeta dx, \tag{3.22}$$

for every $\zeta \in C_c^{1,L}(\overline{\Omega})$. By Theorem 2.4, for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \geq 0$, one has

$$\int_\Omega (|u_n|\, L^*\zeta + \mathrm{sign(u_n)} g(x,u_n)\zeta)\,dx \leq \int_\Omega |\lambda_n|\,\zeta dx. \tag{3.23}$$

We take $\zeta = \eta_1$ as Lemma 2.5, and derive from (3.12),

$$\|u_n\|_{L^1(\Omega)} + \|\rho_{\partial\Omega} g(.,u_n)\|_{L^1(\Omega)} \leq \Theta \int_\Omega \rho_{\partial\Omega} dx + C_1 \|\rho_{\partial\Omega}\lambda_n\|_{L^1(\Omega)}. \tag{3.24}$$

Consequently, by using (3.4) in Proposition 3.2 and (3.9) in Theorem 3.5,

$$\|u_n\|_{M^{(n+\alpha)/(n+\alpha-2)}(\Omega;\rho_{\partial\Omega}^\alpha)} \leq C_2 \|\lambda_n - g(.,u_n)\|_{\mathfrak{M}(\Omega;\rho_{\partial\Omega}^\alpha)} \leq C_3 \left(\Theta + \|\rho_{\partial\Omega}\lambda_n\|_{L^1(\Omega)}\right). \tag{3.25}$$

By the local regularity result of Corollary 2.8, there exist a subsequence $\{u_{n_k}\}$ and a function $u \in W_{loc}^{1,q}(\Omega)$, for any $1 \leq q < n/(n-1)$, such that $u_{n_k} \to u$ a.e. in Ω and weakly in $W_{loc}^{1,q}(\Omega)$. Notice that $W_{loc}^{1,q}(\Omega)$ can be replaced by $W_0^{1,q}(\Omega)$ if $\alpha = 0$, by Theorem 2.7. Combining (3.24) and estimate (2.39) with $\mu_n = 0$ and λ_n replaced by $\lambda_n - g(.,u_n)$, one obtains that, for any Borel subset $\omega \subset \Omega$, there holds

$$\int_\omega |u_n|\,dx \leq \left(C'\,|\Omega| + C_1' \|\rho_{\partial\Omega}\lambda_n\|_{L^1(\Omega)}\right) |\omega|^{1/p},$$

if $p > n$. Thus, by the Vitali Theorem, it can also be assumed that $u_{n_k} \to u$ in $L^1(\Omega)$. Furthermore, for any $R \geq 0$,

$$\begin{aligned}
\int_\omega |g(.,u_n)|\,\rho_{\partial\Omega}^\alpha dx &\leq \int_\omega \tilde{g}(|u_n|)\rho_{\partial\Omega}^\alpha dx \\
&\leq \int_{\omega\cap\{|u_n|\leq R\}} \tilde{g}(|u_n|)\rho_{\partial\Omega}^\alpha dx + \int_{\omega\cap\{|u_n|>R\}} \tilde{g}(|u_n|)\rho_{\partial\Omega}^\alpha dx \\
&\leq \tilde{g}(R)\int_\omega \rho_{\partial\Omega}^\alpha dx - \int_R^\infty \tilde{g}(s)d\theta_n(s),
\end{aligned}$$

where

$$\begin{aligned}
\theta_n(s) = \int_{\{x\in\Omega:|u_n|>s\}} \rho_{\partial\Omega}^\alpha(x)dx &\leq s^{-(n+\alpha)/(n+\alpha-2)} \|u_n\|_{M^{(n+\alpha)/(n+\alpha-2)}(\Omega;\rho_{\partial\Omega}^\alpha)} \\
&\leq Cs^{-(n+\alpha)/(n+\alpha-2)},
\end{aligned}$$

by (3.7). Moreover

$$\begin{aligned}
-\int_R^\infty \tilde{g}(s)d\theta_n(s) &= \tilde{g}(R)\theta_n(R)+\int_R^\infty \theta_n(s)d\tilde{g}(s)\\
&\leq \tilde{g}(R)\theta_n(R)+C\int_R^\infty s^{-(n+\alpha)/(n+\alpha-2)}d\tilde{g}(s)\\
&\leq \tilde{g}(R)\theta_n(R)-C\tilde{g}(R)R^{-(n+\alpha)/(n+\alpha-2)}\\
&\qquad\qquad +\frac{C(n+\alpha)}{n+\alpha-2}\int_R^\infty \tilde{g}(s)s^{-2(n+\alpha-1)/(n+\alpha-2)}ds\\
&\leq \frac{C(n+\alpha)}{n+\alpha-2}\int_R^\infty \tilde{g}(s)s^{-2(n+\alpha-1)/(n+\alpha-2)}ds.
\end{aligned}$$

Since condition (3.9) is equivalent to

$$\int_1^\infty \tilde{g}(s)s^{-2(n+\alpha-1)/(n+\alpha-2)}ds<\infty, \tag{3.26}$$

given $\epsilon>0$, we first choose $R>0$ such that

$$\frac{C(n+\alpha)}{n+\alpha-2}\int_R^\infty \tilde{g}(s)s^{-2(n+\alpha-1)/(n+\alpha-2)}ds\leq\epsilon/2.$$

Then we put $\delta=\epsilon/(2(1+\tilde{g}(R))$ and derive

$$\int_\omega \rho_{\partial\Omega}^\alpha dx\leq\delta\Longrightarrow\int_\omega |g(u_n)|\,\rho_{\partial\Omega}^\alpha dx\leq\epsilon.$$

Therefore $\{\rho_{\partial\Omega}^\alpha g(.,u_n)\}$ is uniformly integrable, and we can assume that the previous sequence $\{n_k\}$ is such that

$$\lim_{n_k\to\infty}\int_\Omega |g_{n_k}(.,u_{n_k})-g(.,u)|\,\rho_{\partial\Omega}^\alpha dx=0\Longrightarrow\int_\Omega |g_{n_k}(.,u_{n_k})-g(.,u)|\,\rho_{\partial\Omega}dx=0, \tag{3.27}$$

since $\alpha\in[0,1]$. Letting $n_k\to\infty$ in (3.22), one obtains

$$\int_\Omega (uL^*\zeta+g(x,u)\zeta)\,dx=\int_\Omega \zeta d\lambda. \tag{3.28}$$

□

Since the uniform integrability conditions depends only on the total variation norm of the measure $\rho_{\partial\Omega}^\alpha\lambda$, the following stability result holds.

Corollary 3.8 *Let Ω and α be as in Theorem 3.7, g satisfy the (n,α)-weak-singularity assumption and $r\mapsto g(x,r)$ is nondecreasing, for any $x\in\Omega$. Then the solution u is unique. If we assume that $\{\lambda_m\}$ is a sequence of measures in $\mathfrak{M}(\Omega;\rho_{\partial\Omega}^\alpha)$ such that*

$$\lim_{m\to\infty}\int_\Omega \zeta d\lambda_m=\lim_{m\to\infty}\int_\Omega \zeta d\lambda,$$

for any $\zeta \in C(\overline{\Omega})$ *satisfying* $\sup_\Omega \rho_{\partial\Omega}^{-\alpha} |\zeta| < \infty$, *then the corresponding solutions* u_m *of problem*

$$\begin{aligned} Lu_m + g(x, u_m) &= \lambda_m \quad \text{in } \Omega, \\ u_m &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.29}$$

converge in $L^1(\Omega)$ *to the solution* u *of (3.1), when* $m \to \infty$.

Remark. If $g(x, r) = |r|^{q-1} r$, the (n, α)-weak-singularity assumption is satisfied if and only if

$$0 < q < \frac{n + \alpha}{n + \alpha - 2}. \tag{3.30}$$

3.2 Admissible measures and the Δ_2-condition

Definition 3.9 Let $\tilde{g}$ be a continuous real valued nondecreasing function defined in $\mathbb{R}_+$, $\tilde{g} \geq 0$. A measure λ in Ω is said $(\tilde{g}, k)$-admissible if

$$\int_\Omega \tilde{g}(\mathbb{G}_L^\Omega(|\lambda|) + k)\rho_{\partial\Omega} dx < \infty, \tag{3.31}$$

where $G_L^\Omega(|\lambda|)$ is the Green potential of λ and $k \geq 0$.

Theorem 3.10 *Let* Ω *be a* C^2 *bounded domain in* $\mathbb{R}^n$, $n \geq 2$, L *an elliptic operator defined by (2.1), and* $g \in C(\Omega \times \mathbb{R})$. *We assume that* L *satisfies the condition (H), and* g *(3.12) for some* $r_0 \geq 0$ *and (3.14) for some function* $\tilde{g}$ *as in Definition 3.9. Then for any* $(\tilde{g}, r_0)$*-admissible Radon measure* $\lambda \in \mathfrak{M}(\Omega; \rho_{\partial\Omega})$, *Problem (3.1) admits a solution.*

Proof. For $k > 0$, we take the same truncation $g_k(., r)$ of $g(., r)$ defined by (3.15). Since g_k satisfies (3.13) and (3.14), we denote by u_k a solution of

$$\begin{aligned} Lu_k + g_k(x, u_k) &= \lambda \quad \text{in } \Omega, \\ u_k &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.32}$$

which exists by Theorem 3.7. As in the proof of Theorem 3.7 the following estimates hold,

$$\|u_k\|_{L^1(\Omega)} + \|\rho_{\partial\Omega} g_k(., u_k)\|_{L^1(\Omega)} \leq \Theta \int_\Omega \rho_{\partial\Omega} dx + C_1 \|\rho_{\partial\Omega}\lambda\|_{\mathfrak{M}(\Omega)}, \tag{3.33}$$

where Θ is defined by (3.18), and

$$\|u_k\|_{M^{(n+1)/(n-1)}(\Omega; \rho_{\partial\Omega})} \leq C_3 \left(\Theta + \|\rho_{\partial\Omega}\lambda\|_{L^1(\Omega)}\right). \tag{3.34}$$

By Corollary 2.8, there exist a subsequence $\{u_{k_j}\}$ and a function $u \in W^{1,q}_{loc}(\Omega)$, for any $1 \leq q < n/(n-1)$, such that $u_{k_j} \to u$ a.e. in Ω and weakly in $W^{1,q}_{loc}(\Omega)$. Moreover $g_{k_j}(., u_{k_j}) \to g(., u)$ almost everywhere in Ω. Put

$$w_{\lambda_+} = \mathbb{G}_L^\Omega(\lambda_+) + r_0.$$

Then

$$L(u_k - w_{\lambda_+}) + g_k(x, u_k) = \lambda - \lambda_+,$$

and, for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \geq 0$,

$$\begin{aligned}\int_\Omega (u_k - w_{\lambda_+})_+ L^*\zeta dx + \int_\Omega g_k(x, u_k)\mathrm{sign}_+(\mathrm{u_k} - \mathrm{w}_{\lambda_+})\zeta \mathrm{dx} \\ \leq -\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}}(u_k - w_{\lambda_+})_+ dS,\end{aligned} \tag{3.35}$$

by Inequality (2.40). Since the boundary term in (3.35) vanishes, and $w_{\lambda_+} \geq r_0$, there holds $g_k(x, u_k)\mathrm{sign}_+(\mathrm{u_k} - \mathrm{w}_{\lambda_+}) \geq 0$, which implies

$$\int_\Omega (u_k - w_{\lambda_+})_+ L^*\zeta dx \leq 0.$$

Taking $\zeta = \eta_1$ defined by (2.24) (with $u = 1$, hence $L^*\eta_1 = 1$), yields to $(u_k - w_{\lambda_+})_+ = 0$ a.e. in Ω. Thus

$$u_k \leq w_{\lambda_+} = \mathbb{G}_L^\Omega(\lambda_+) + r_0.$$

In the same way

$$u_k \geq -\mathbb{G}_L^\Omega(\lambda_-) - r_0.$$

Therefore

$$|u_k| \leq \mathbb{G}_L^\Omega(|\lambda|) + r_0 \Longrightarrow |g_k(u_k)| \leq \tilde{g}(|u_k|) \leq \tilde{g}(\mathbb{G}_L^\Omega(|\lambda|) + r_0). \tag{3.36}$$

Because the right-hand side of (3.36) belongs to $L^1(\Omega; \rho_{\partial\Omega}dx)$, the sequence $\{g_k(., u_k)\}$ is uniformly integrable for the measure $\rho_{\partial\Omega}dx$. As in the proof of Theorem 3.7, we conclude by the Vitali Theorem that u is a solution of (3.1). $\square$

The condition of (g, r_0)-admissibility on λ is too restrictive if the function g has a strong power growth, in particular it leads to exclude some λ which are regular with respect the n-dimensional Hausdorff measure, even if we know, from the Brezis and Strauss Theorem (see Theorem 3.7-Step 1), that Problem (3.1) is solvable for such measures. A natural extension is to impose only the (g, r_0)-admissibility on the singular part λ_s of the measure. However, a generic power-like growth condition called the Δ_2-condition is needed.

Definition 3.11 A real valued function $g \in C(\Omega \times \mathbb{R})$ satisfies a uniform Δ_2-condition if there exist two constants $\ell \geq 0$, $\theta > 1$ such that

$$\left|g(x, r + r')\right| \leq \theta(|g(x, r)| + \left|g(x, r')\right|) + \ell, \quad \forall x \in \Omega, \forall (r, r') \in \mathbb{R} \times \mathbb{R}. \tag{3.37}$$

Theorem 3.12 *Let Ω and L be as in Theorem 3.10. Assume $g \in C(\Omega \times \mathbb{R})$ satisfies the Δ_2-condition, $r \mapsto g(x, r)$ is nondecreasing for any $x \in \Omega$ and (3.14) holds for some function $\tilde{g}$ as in Definition 3.9. For any Radon measure $\lambda \in \mathfrak{M}(\Omega; \rho_{\partial\Omega})$, with $\lambda = \tilde{\lambda} + \lambda^*$, where $\tilde{\lambda} \in L^1(\Omega; \rho_{\partial\Omega}dx)$, and λ^* is $(\tilde{g}, 0)$-admissible and singular with respect to the n-dimensional Lebesgue measure, Problem (3.1) admits a unique solution.*

Proof. Uniqueness comes from the monotonicity of $r \mapsto g(x,r)$.

Step 1 If we write $g(x,r) = g(x,r) - g(x,0) + g(x,0) = \hat{g}(x,r) + g(x,0)$, then the equation is transformed into

$$Lu + \hat{g}(x,u) = \lambda - g(x,0) = \hat{\lambda},$$

where $r \mapsto \hat{g}(x,r)$ nondecreasing and $\hat{g}(x,0) = 0$. Notice that $|\hat{g}(x,0)| \leq \tilde{g}(0)$ by (3.14), and that λ^* is singular with respect to $\tilde{\lambda} - g(x,0)$. Finally the new function $\hat{g}$ satisfies Inequality (3.37) with the same θ and ℓ replaced by $\hat{\ell} = \ell + (2\theta + 1)\,|\tilde{g}(0)|$, and (3.14) with $\tilde{g}$ replaced by $\tilde{g} + |\tilde{g}(0)|$. From now we shall suppose that the function g satisfies $g(x,0) = 0$ for any $x \in \Omega$. We introduce the truncation $g_k(.,r)$ by (3.15). The truncated function g_k satisfies also (3.37) (with θ replaced by $1 + \theta$).

Step 2 We suppose that λ is nonnegative. Then $\tilde{\lambda}$ and λ^* inherit the same property. Let $\{\tilde{\lambda}_i\}$ be a sequence of smooth nonnegative functions with compact support in Ω, converging to $\tilde{\lambda}$ in the weak sense of $L^1(\Omega;\rho_{\partial\Omega})$. Let $u_{i,k}$ be the solution of

$$\begin{aligned} Lu_{i,k} + g_k(x,u_{i,k}) &= \tilde{\lambda}_i + \lambda^* && \text{in } \Omega, \\ u_{i,k} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.38}$$

and $v_{i,k}$ the one of

$$\begin{aligned} Lv_{i,k} + g_k(x,v_{i,k}) &= \tilde{\lambda}_i && \text{in } \Omega, \\ v_{i,k} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.39}$$

Both solutions exist by Theorem 3.10. By the maximum principle

$$0 \leq u_{i,k} \leq v_{i,k} + \mathbb{G}_L^{\Omega}(\lambda^*), \tag{3.40}$$

and by the monotonicity of g_k and (3.37),

$$0 \leq g_k(.,u_{i,k}) \leq \theta\left(g_k(.,v_{i,k}) + g_k(.,\mathbb{G}_L^{\Omega}(\lambda^*))\right) + \ell \leq \theta\left(g_k(.,v_{i,k}) + \tilde{g}(\mathbb{G}_L^{\Omega}(\lambda^*))\right) + \ell. \tag{3.41}$$

By Theorem 3.10), if i is fixed and $k \to \infty$, the sequence $\{v_{i,k}\}$ converges weakly in $W^{1,q}_{loc}(\Omega)$ and a.e. in Ω to the solution v_i of

$$\begin{aligned} Lv_i + g(x,v_i) &= \tilde{\lambda}_i && \text{in } \Omega, \\ v_i &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.42}$$

Since the $v_{i,k}$ are uniformly bounded with respect to k, the same property holds with the $g_k(v_{i,k})$, hence their convergence to v_i and $g(.,v_i)$ are uniform in $\overline{\Omega}$. Because of (3.41) and the elliptic equations regularity theory, the sequence $\{u_{i,k}\}_{k\in\mathbb{N}_*}$ is relatively compact in the $W^{1,q}_{loc}(\Omega)$-topology. Thus there exist a subsequence $\{u_{i,k_j}\}$ and a function u_i such that $u_{i,k_j} \to u_i$ as $k_j \to \infty$ in this topology and a.e. in Ω. By continuity, $g_{k_j}(.,u_{i,k_j}) \to g(.,u_i)$ a.e. in Ω. Because of (3.41) and the $(\tilde{g},0)$-admissibility condition on λ^*, Lebesgue's theorem implies that

$$\lim_{k_j\to\infty} g_{k_j}(.,u_{i,k_j}) = g(.,u_i) \quad \text{in } L^1(\Omega;\rho_{\partial\Omega}dx).$$

It follows from inequality (3.40) that $u_{i,k_j} \to u_i$ in $L^1(\Omega)$ (we recall that $\mathbb{G}_L^\Omega(\lambda^*) \in L^1(\Omega)$). Letting $k_j \to \infty$ in (3.38) we see that u_i is the solution of

$$\begin{aligned} Lu_i + g(x,u_i) &= \tilde{\lambda}_i + \lambda^* &&\text{in } \Omega, \\ u_i &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{3.43}$$

By uniqueness of u_i the whole sequence $u_{i,k}$ converges to u_i as $k \to \infty$. Moreover

$$\begin{aligned} &(i)\, 0 \le u_i \le v_i + \mathbb{G}_L^\Omega(\lambda^*), \\ &(ii)\, 0 \le g(.,u_i) \le \theta\left(g(.,v_i) + g(\mathbb{G}_L^\Omega(\lambda^*))\right) + \ell \le \theta\left(g(.,v_i) + \tilde{g}(\mathbb{G}_L^\Omega(\lambda^*))\right) + \ell. \end{aligned} \tag{3.44}$$

By Theorem 2.4 with $\zeta = G_L^\Omega(1)$,

$$\|v_i - v_j\|_{L^1(\Omega)} + \|g(.,v_i) - g(.,v_j)\|_{L^1(\Omega;\rho_{\partial\Omega}dx)} \le C\left\|\tilde{\lambda}_i - \tilde{\lambda}_j\right\|_{L^1(\Omega)}. \tag{3.45}$$

Therefore $v_i \to v$ in $L^1(\Omega)$ and $g(.,v_i) \to g(.,v)$ in $L^1(\Omega;\rho_{\partial\Omega}dx)$ where v is the solution of

$$\begin{aligned} Lv + g(x,v) &= \tilde{\lambda} &&\text{in } \Omega, \\ v &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{3.46}$$

By (3.44)-(i) there exists a subsequence $\{u_{i_j}\}$ which converges in $L^1(\Omega)$ and a.e. in Ω to some function u. Because of (3.44)-(ii), the admissibility condition on λ^* and the Vitali Theorem, the sequence $\{g(.,u_{i_j})\}$ converges to $g(.,u)$ in $L^1(\Omega;\rho_{\partial\Omega}dx)$. Thus u is the solution of (3.1).

Step 3 In the general case we construct the solution $u_{i,k}$ of (3.38) and the functions $U = \overline{u}_{i,k}$ and $U = \underline{u}_{i,k}$ solutions of

$$\begin{aligned} LU + g_k(x,U) &= \tilde{\Lambda} &&\text{in } \Omega, \\ U &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{3.47}$$

where $\Lambda = \left|\tilde{\lambda}_i\right| + |\lambda^*|$ in the case of $\overline{u}_{i,k}$ and $\Lambda = -\left|\tilde{\lambda}_i\right| - |\lambda^*|$ in the case of $\underline{u}_{i,k}$. We also construct the solutions $V = \overline{v}_{i,k}$ and $V = \underline{v}_{i,k}$ of the same equation with $\Lambda = \left|\tilde{\lambda}_i\right|$ in the case of $\overline{v}_{i,k}$ and $\Lambda = -\left|\tilde{\lambda}_i\right|$ in the case of $\underline{v}_{i,k}$. Since

$$\underline{v}_{i,k} - \mathbb{G}_L^\Omega(|\lambda^*|) \le u_{i,k} \le \overline{v}_{i,k} + \mathbb{G}_L^\Omega(|\lambda^*|), \tag{3.48}$$

and

$$\theta\left(g_k(.,\underline{v}_{i,k}) + g(.,\mathbb{G}_L^\Omega(-|\lambda^*|))\right) - \ell \le g_k(.,u_{i,k}) \le \theta\left(g_k(.,\overline{v}_{i,k}) + g(.,\mathbb{G}_L^\Omega(|\lambda^*|))\right) + \ell, \tag{3.49}$$

we conclude by using the Vitali Theorem and the convergence arguments of Step 2. □

3.3 The duality method

Let Ω be a domain in $\mathbb{R}^n$ and L is an elliptic operator in Ω. In this section we study the sharp solvability of Problem (3.1) when $g(x,r) = |u|^{q-1}u$ with $q > 0$. For this type of nonlinearity, the $(n,0)$-weak-singularity assumption is satisfied if and only if $0 < q < n/(n-2)$. Thus we shall concentrate on the case $n \geq 3$ and $q \geq n/(n-2)$ and for such a task the theory of Bessel capacities is needed.

3.3.1 Bessel capacities

Let $p > 1$ be a real number and $p' = p/(p-1)$. If m in an integer we endow the Sobolev space $W^{m,p}(\mathbb{R}^n)$ with the usual norm

$$\|\phi\|_{W^{m,p}(\mathbb{R}^n)} = \left(\sum_{|\gamma|\leq m} \int_\Omega |D^\gamma \phi|^p dx\right)^{1/p},$$

and we introduce the associated capacity $C_{m,p}$ by

$$C_{m,p}(K) = \inf\left\{\|\phi\|^p_{W^{m,p}(\mathbb{R}^n)} : \phi \in C_c^\infty(\mathbb{R}^n),\ \phi \geq 1 \text{ in a neighborhood of } K\right\},$$

if K is compact,

$$C_{m,p}(G) = \sup\left\{C_{m,p}(K) : K \subset G,\ K \text{ compact }\right\},$$

if G is open, and

$$C_{m,p}(E) = \inf\left\{C_{m,p}(G) : E \subset G,\ G \text{ open }\right\},$$

for an arbitrary set E. The scale of Sobolev spaces is not accurate enough to describe the subets of $\mathbb{R}^n$ by means of their capacities. If α is a real number, we introduce the Bessel kernel of order α by

$$G_\alpha = \mathcal{F}^{-1}\left((1+|\xi|^2)^{-\alpha/2}\right) \tag{3.50}$$

were $\mathcal{F}^{-1}$ is the inverse Fourier transform on the Schwartz space $\mathcal{S}'(\mathbb{R}^n)$. If

$$\mathcal{G}_\alpha = (I-\Delta)^{-\alpha/2},$$

there holds the Bessel potential representation

$$f = \mathcal{G}_\alpha g = G_\alpha * g \Longleftrightarrow g = \mathcal{G}_{-\alpha} g = G_{-\alpha} * f \quad \forall f,\, g \in \mathcal{S}(\mathbb{R}^n). \tag{3.51}$$

Definition 3.13 Let α and $p > 1$ be two real numbers. The Bessel potential space of order α and power p is

$$L^{\alpha,p}(\mathbb{R}^n) = \left\{f : f = G_\alpha * g,\ g \in L^p(\mathbb{R}^n)\right\},$$

with norm

$$\|f\|_{L^{\alpha,p}(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)} = \|G_{-\alpha} * f\|_{L^p(\mathbb{R}^n)}.$$

As usual, $L_0^{\alpha,p'}(\mathbb{R}^n)$ denotes the closure of $C_c^\infty(\mathbb{R}^n)$ in $L^{\alpha,p'}(\mathbb{R}^n)$. Thanks to a result due to Calderon, the functions in $W^{m,p}(\mathbb{R}^n)$ can be represented by mean of Bessel potentials. Actually for any $\alpha \in \mathbb{N}_*$ and $1 < p < \infty$, $W^{\alpha,p}(\mathbb{R}^n) = L^{\alpha,p}(\mathbb{R}^n)$ and their exists a positive constant A such that

$$A^{-1}\|f\|_{L^{\alpha,p}(\mathbb{R}^n)} \le \|f\|_{W^{\alpha,p}(\mathbb{R}^n)} \le A\|f\|_{L^{\alpha,p}(\mathbb{R}^n)}, \quad \forall f \in W^{\alpha,p}(\mathbb{R}^n). \tag{3.52}$$

By generalization (see [28] for a general construction of capacities), the Bessel capacity of order (α, p) $(\alpha > 0,\ p > 1)$ of a compact set K is defined by

$$C_{\alpha,p}(K) = \inf \left\{ \|\phi\|^p_{L^{\alpha,p}(\mathbb{R}^n)} : \phi \in \mathcal{S}(\mathbb{R}^n),\ \phi \ge 1 \text{ in a neighborhood of } K \right\}, \tag{3.53}$$

with the same extension to open sets and arbitrary sets as for Sobolev capacities. A dual definition involving measures is the following [1] :

$$C_{\alpha,p}(K) = \sup \left\{ \left(\frac{\mu(K)}{\|G_\alpha * \mu\|_{L^{p'}(\mathbb{R}^n)}} \right)^p : \mu \in \mathfrak{M}_+(K) \right\}, \tag{3.54}$$

where $\mathfrak{M}_+(K)$ is the set of positive Radon measures with support in K. An important result due to Maz'ya (see [1]) states that the following expression

$$\tilde{C}_{\alpha,p}(K) = \inf \left\{ \|\phi\|^p_{L^{\alpha,p}(\mathbb{R}^n)} : \phi \in \mathcal{S}(\mathbb{R}^n),\ \phi \equiv 1 \text{ in a neighborhood of } K \right\}, \tag{3.55}$$

defines a new capacity which is equivalent to the $C_{\alpha,p}$-capacity in the sense that there exists a positive constant B such that

$$B^{-1}C_{\alpha,p}(K) \le \tilde{C}_{\alpha,p}(K) \le BC_{\alpha,p}(K),$$

for any compact subset K. In the particular case of sets with zero capacity, the following useful result holds.

Proposition 3.14 *Let $\alpha > 0$, $1 < p < \infty$, K be a compact subset of $\mathbb{R}^n$ and $\mathcal{O}$ an open subset containing K. If $C_{\alpha,p}(K) = 0$, there exists a sequence $\{\phi_n\} \subset C_c^\infty(\mathcal{O})$ such that $0 \le \phi_n \le 1$, $\phi_n \equiv 1$ in a neighborhood of K and $\phi_n \to 0$ in $L^{\alpha,p}(\mathbb{R}^n)$ as $n \to \infty$.*

By using smooth cut-off function with value in $[0, 1]$, support in a neighborhood of K and taking the value 1 in a smaller neighborhood of K, the proof of this result is straightforward if α is an integer, and more delicate if not (see [1, Th. 3.3.3]).

Definition 3.15 Let $\alpha > 0$ and $1 < p < \infty$.

(i) A property is said to hold $C_{\alpha,p}$-quasi everywhere if it holds everywhere but on a set of $C_{\alpha,p}$-capacity zero.

(ii) A function ϕ defined in $\mathbb{R}^n$ is said to be $C_{\alpha,p}$-quasicontinuous if for any $\epsilon > 0$, there is an open set $G \subset \mathbb{R}^n$ with $C_{\alpha,p}(G) < \epsilon$ and $f \in C(G^c)$.

(iii) Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and $\lambda \in \mathfrak{M}(\mathcal{O})$. The measure λ is said not to charge subsets of $\mathcal{O}$ with $C_{\alpha,p}$-capacity zero if

$$\forall E \subset \mathcal{O}, C_{\alpha,p}(E) = 0 \Longrightarrow \int_E d|\lambda| = 0,$$

where, $d|\lambda|$ denote in the same way the unique complete regular Borel measure generated by the Radon measure $|\lambda|$.

It is proven in [1] that for any $\alpha > 0$, $1 < p < \infty$ and $g \in L^p(\Omega)$, the function $G_\alpha * g$ is $C_{\alpha,p}$-quasicontinuous. Therefore, any element $\phi \in L^{\alpha,p}(\mathbb{R}^n)$ admits a (unique) quasicontinuous representative, $\tilde{\phi}$. Furthermore, from any converging sequence $\{\phi_n\} \subset L^{\alpha,p}(\mathbb{R}^n)$ it can be extracted a subsequence $\{\phi_n\}$ which converges $C_{\alpha,p}$-quasi everywhere. The link between the measures which do not charge capacitary sets and elements of negative Bessel spaces is enlighted by three results. The first one is due essentially to Grun-Rehomme [50] (see also [1]).

Proposition 3.16 *Let $\alpha > 0$ and $1 < p < \infty$. If $\lambda \in \mathfrak{M}(\Omega) \cap L^{-\alpha,p}(\Omega)$, then λ does not charge sets with $C_{\alpha,p'}$-capacity zero.*

Proof. By the Jordan decomposition Theorem of a measure, there exist two disjoint Borel subsets A and B such that

$$A \cup B = \Omega, \quad \lambda_+(B) = 0, \quad \lambda_-(A) = 0.$$

Let $E \subset \mathbb{R}^n$ with $C_{\alpha,p'}(E) = 0$. With no loss of generality E can be assumed as being a Borel set. It is therefore sufficient that $\lambda_+(A \cap E) = \lambda_-(B \cap E) = 0$. Because

$$\lambda_+(A \cap E) = \sup\{\lambda_+(K) : K \text{ compact }, k \subset A \cup E\},$$

it is sufficient to prove that for any compact subset $K \subset A \cap E$, $\lambda_+(K) = 0$. Let $\epsilon > 0$, since $\lambda_-(K) = 0$, there exists an open subset ω of Ω containing K such that $\lambda_-(\omega) \leq \epsilon$. Let $\eta \in C_c^\infty(\omega)$, with value in $[0, 1]$ and equal to 1 on K. By Proposition 3.14, since $C_{\alpha,p'}(K) = 0$, there exists a sequence $\{\phi_n\} \subset C_c^\infty(\Omega)$, of functions with value in $[0, 1]$, equal to 1 in a neighborhood of K and such that $\phi_n \to 0$ in $L^{\alpha,p'}(\Omega)$ as $n \to \infty$. Then

$$\int_K d\lambda_+ \leq \int_K \phi_n \eta d\lambda_+ \leq \int_\gamma \phi_n \eta d\lambda_+ = \int_\Omega \phi_n \eta d\lambda + \int_\omega \phi_n \eta d\lambda_-.$$

But

$$\int_\omega \phi_n \eta d\lambda_- \leq \int_\omega d\lambda_- \leq \epsilon,$$

and

$$\int_\Omega \phi_n \eta d\lambda \leq \int_\Omega \phi_n d\lambda = \langle \lambda, \phi_n \rangle_{[L^{-\alpha,p}, L^{\alpha,p'}]} \leq \|\lambda\|_{L^{-\alpha,p}} \|\phi\|_{n L^{\alpha,p'}},$$

which goes to zero as $n \to \infty$. Therefore

$$\int_K d\lambda_+ \leq \epsilon.$$

Since ϵ is arbitrary, $\lambda_+(K) = 0$. In the same way $\lambda_-(B \cap E) = 0$. Therefore $|\lambda|(E) = 0$. $\square$

The second result is due to Feyel and de la Pradelle [42]. It shows the constructivity of certain measures which do not charge sets a given capacity of which vanishes.

Proposition 3.17 *Let $\alpha > 0$ and $1 < p < \infty$. If $\lambda \in \mathfrak{M}_+(\Omega)$ does not charge sets with $C_{\alpha,p'}$-capacity zero, there exists an increasing sequence $\{\lambda_n\} \subset \mathfrak{M}^b_+(\Omega) \cap L^{-\alpha,p}(\Omega)$, λ_n with compact support in Ω, which converges to λ.*

Proof. We first assume that λ has compact support in Ω. Let $\phi \in L_0^{\alpha,p'}(\Omega)$ and $\tilde{\phi}$ its quasicontinuous representative. Since the function $\tilde{\phi}_+$ is quasicontinuous too, the following functional is well defined on $L_0^{\alpha,p'}(\Omega)$, with values in $[0,\infty]$,

$$P(\phi) = \int_\Omega \tilde{\phi}_+ d\lambda. \tag{3.56}$$

If $\{\phi_n\}$ converges to ϕ in $L_0^{\alpha,p'}(\Omega)$, there exists a subsequence $\{\phi_{n_k}\}$ which converges $C_{\alpha,p'}$-quasi everywhere. Hence

$$\int_\Omega \tilde{\phi}_+ d\lambda \leq \liminf_{n_k \to \infty} \int_\Omega \tilde{\phi}_{n\,+} d\lambda,$$

by Fatou's lemma, and $\phi \mapsto P(\phi)$ is lower semicontinuous. Since P is convex and positively homogeneous of order 1, it is the upper hull of all the continuous linear functionals it dominates, by the Hahn-Banach Theorem.

Step 1 Let $\epsilon > 0$, and $\phi_0 \in L_0^{\alpha,p'}(\Omega)$. Then we claim that there exists a positive Radon measure θ belonging to $L^{-\alpha,p}(\Omega)$ such that $0 \leq \theta \leq \lambda$, and

$$\int_\Omega \phi_0 d(\nu - \theta) < \epsilon. \tag{3.57}$$

Clearly

$$(\phi_0, P(\phi_0) - \epsilon) \notin Epi(P) = \left\{(\phi, t) \in L_0^{\alpha,p'}(\Omega) \times \mathbb{R} : t \geq P(\phi)\right\}.$$

Since $Epi(P)$ is a closed convex subset of $L_0^{\alpha,p'}(\Omega) \times \mathbb{R}$, it follows by the Hahn-Banach Theorem that there exist a continuous form ℓ on $L_0^{\alpha,p'}(\Omega)$ and two constants a and b such that

$$a + bt + \ell(\phi) \leq 0, \quad \forall (\phi, t) \in Epi(P), \tag{3.58}$$

and

$$a + b(P(\phi_0) - \epsilon) + \ell(\phi_0) > 0. \tag{3.59}$$

But $(0,0) \in Epi(P) \Longrightarrow a \leq 0$. Thus (3.59) holds with $a = 0$. If we apply (3.58) to $(\tau\phi, \tau t)$ with $\tau > 0$ arbitrary (such a couple belongs to $Epi(P)$ since P is positively homogeneous) and let $\tau \to \infty$, it follows

$$bt + \ell(\phi) \leq 0, \quad \forall (\phi, t) \in Epi(P). \tag{3.60}$$

In the particular case $\phi = 0$ and $t > 0$ (possible since $(0,t) \in Epi(P)$, $\forall t > 0$), it gives $b \leq 0$. If b were zero one would have $\ell(\phi) \leq 0$ for any $(\phi, t) \in Epi(P)$, and in particular $\ell(\phi_0) \leq 0$, which would contradict (3.59) if we impose $b = 0$. Since $b < 0$, we define θ by

$$\theta(\phi) = -\frac{\ell(\phi)}{b}, \quad \forall \phi \in L_0^{\alpha,p'}(\Omega),$$

and derive

$$P(\phi) \geq \theta(\phi), \tag{3.61}$$

for any $\phi \in L_0^{\alpha,p'}(\Omega)$, since $(P(\phi), \phi) \in Epi(P)$. In the particular case where $\phi \leq 0$, there holds $\theta(\phi) \leq 0$. This means that θ is a continuous positive linear functional on $L_0^{\alpha,p'}(\Omega)$, dominated by P. It defines a unique Radon measure, still denoted by θ, and (3.57) holds.

Step 2 End of the proof. We assume now that λ has no longer a compact support in Ω. There exists an exhaustive sequence of open subsets $\{\Omega_k\}$, compactely included in Ω such that

$$\Omega_k \subset \overline{\Omega}_k \subset \Omega_{k+1} \subset \overline{\Omega}_{k+1} \subset \ldots \Omega.$$

We put $\lambda_k = \lambda|_{\Omega_k}$. We apply the result of step 1 to λ_k, with $\epsilon = 1/k$ and $\phi \equiv 1$ on Ω_k and derive the existence of a positive Radon measure $\theta_k \in L^{\alpha,p'}(\Omega)$, with compact support in Ω satisfying $0 \leq \theta_k \leq \lambda$ and

$$\int_{\Omega_k} d(\lambda - \theta_k) < 1/k.$$

The measure $\lambda_n = \sup\{\theta_1, \theta_2, \ldots, \theta_n\}$ has compact support in Ω, $\lambda_n \leq \lambda_{n+1} \leq \lambda$ for any n, and

$$\lim_{n\to\infty} \int_\Omega \zeta d\lambda_m = \int_\Omega \zeta d\lambda, \quad \forall \zeta \in C_c(\Omega).$$

$\square$

Corollary 3.18 *Let $\alpha > 0$ and $1 < p < \infty$. If $\lambda \in \mathfrak{M}^b(\Omega)$ does not charge sets with $C_{\alpha,p'}$-capacity zero, there exist a function $\lambda^* \in L^1(\Omega)$ and a measure $\tilde{\lambda} \in L^{-\alpha,p}(\Omega)$ such that*

$$\lambda = \tilde{\lambda} + \lambda^*. \tag{3.62}$$

Proof. By assumption, both the positive and the negative parts of λ do not charge sets with $C_{\alpha,p'}$-capacity zero. Therefore it is sufficient to prove (3.62) with $\lambda \in \mathfrak{M}_b^+(\Omega)$. Let $\{\lambda_n\} \subset L^{-\alpha,p}(\Omega) \cap \mathfrak{M}_+(\Omega)$ be the increasing sequence of measures with compact support in Ω which converges to λ weakly. We set

$$\rho_j = \lambda_j - \lambda_{j-1}, \quad \text{for } j \in \mathbb{N}_*, \quad \text{and } \rho_0 = \lambda_0.$$

Then

$$\lambda = \sum_{j=0}^{\infty} \rho_j,$$

and the series converges strongly in the space $\mathfrak{M}^b(\Omega)$. In particular

$$\sum_{j=0}^{\infty} \|\rho_j\|_{\mathfrak{M}^b(\Omega)} < \infty.$$

Let $\{\eta_k\}_{k\in\mathbb{N}_*}$ be a sequence of C^∞ nonnegative functions in $\mathbb{R}^n$, with compact support in the open ball $B_{k^{-1}}(0)$, satisfying

$$\int_\Omega \eta_k dx = 1.$$

For any $j \in \mathbb{N}_*$ there exists $k_j^0 \in \mathbb{N}_*$ such that for $k \geq k_j^0$, $\rho_{j,k} = \rho_j * \eta_k \in C_c^\infty(\Omega)$. Since $\rho_{j,k} \to \rho_j$ as $k \to \infty$, we fix $k_j \geq k_j^0$ such that

$$\left\|\rho_{j,k_j} - \rho_j\right\|_{L^{-\alpha,p}(\Omega)} \leq 2^{-j}.$$

We set $\tilde{\rho}_{j,k_j} = \rho_j - \rho_{j,k_j}$. The series $\sum_{j=0}^{\infty} \tilde{\rho}_{j,k_j}$ is normaly convergent in $L^{-\alpha,p}(\Omega)$ and, if $\tilde{\lambda}$ denotes its sum, it belongs to $L^{-\alpha,p}(\Omega)$. Moreover

$$\left\|\rho_{j,k_j}\right\|_{L^1(\Omega)} = \left\|\rho_j * \eta_{k_j}\right\|_{L^1(\Omega)} = \|\rho_j\|_{\mathfrak{M}^b(\Omega)}.$$

Thus the series $\sum_{j=0}^{\infty} \rho_{j,k_j}$ is normaly convergent in $L^1(\Omega)$ with sum λ^*. The three series $\sum_{j=0}^{\infty} \rho_j$, $\sum_{j=0}^{\infty} \tilde{\rho}_{j,k_j}$ and $\sum_{j=0}^{\infty} \rho_{j,k_j}$ converge in the sense of distributions in Ω, therefore (3.62) holds. $\square$

Remark. If $\lambda \geq 0$, it is the same with λ^*. Unfortunately it is not clear that $\tilde{\lambda}$ inherits the same property. Notice that λ^* and $\tilde{\lambda}$ may not be mutually singular.

Another important and useful result concerning measures which do not charge sets with zero capacity is the following [29].

Theorem 3.19 *Let $\alpha > 0$ and $1 < p < \infty$. If $\lambda \in \mathfrak{M}_+(\Omega)$ does not charge sets with $C_{\alpha,p'}$-capacity zero, there exist $\nu \in \mathfrak{M}_+(\Omega) \cap L^{-\alpha,p}(\Omega)$ and a Borel function f with value in $[0,\infty)$ such that*

$$\lambda(E) = \int_E f d\nu, \quad \forall E \subset \Omega, \; E \text{ Borel.} \tag{3.63}$$

3.3.2 Sharp solvability

The following theorem due to Baras and Pierre [9] characterizes the bounded measures for which the problem

$$\begin{aligned} Lu + |u|^{q-1} u &= \lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.64}$$

admits a solution.

Theorem 3.20 *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 3$, L the elliptic operator defined by (2.1) satisfying the condition (H), $q \geq n/(n-2)$ and $\lambda \in \mathfrak{M}^b(\Omega)$. Then Problem (3.64) admits a solution if and only if λ does not charge sets with $C_{2,q'}$-capacity zero. The solution is unique and the mapping $\lambda \mapsto u$ is nondecreasing.*

For proving this theorem we need the following regularity result.

Lemma 3.21 *Let Ω and L be as in Theorem 3.20. Then for any $1 < p < \infty$ and $\lambda \in W^{-2,p}(\Omega) \cap \mathfrak{M}^b(\Omega)$, $\mathbb{G}_L^\Omega(\lambda) \in L^p(\Omega)$. Moreover there exists $C = C(\Omega, L, p) > 0$ such that*

$$\left\|\mathbb{G}_L^\Omega(\lambda)\right\|_{L^p(\Omega)} \leq C\|\lambda\|_{W^{-2,p}(\Omega)}. \tag{3.65}$$

Proof. Put $v = \mathbb{G}_L^\Omega(\lambda)$, then

$$\int_\Omega vL^*\zeta dx = \int_\Omega \zeta d\lambda, \quad \forall \zeta \in C_c^{1,L}(\overline{\Omega}).$$

Let $\phi \in C_0^\infty(\Omega)$, $\zeta = \mathbb{G}_{L^*}^\Omega(\phi)$, then

$$\left|\int_\Omega v\phi dx\right| \leq \|\lambda\|_{W^{-2,p}(\Omega)}\|\zeta\|_{W^{2,p'}(\Omega)} \leq C\|\lambda\|_{W^{-2,p}(\Omega)}\|\phi\|_{L^{p'}(\Omega)},$$

by the L^p-regularity theory of elliptic equations. Hence $v \in L^p(\Omega)$ and (3.65) follows. □

Proof of Theorem 3.20. (i) Assume that u is a solution of (3.64). Since $|u|^{q-1}u \in L^1(\Omega)$ by Proposition 3.2, it does not charge set with $C_{2,q'}$-capacity zero, which are negligible sets for the n-dimensional Hausdorff measure. Therefore $Lu \in \mathfrak{M}^b(\Omega)$, and

$$|\langle Lu, \phi\rangle| = \left|\int_\Omega uL^*\phi dx\right| \leq \|u\|_{L^q(\Omega)}\|L^*\phi\|_{L^{q'}(\Omega)} \leq C\|u\|_{L^q(\Omega)}\|\phi\|_{W^{2,q'}(\Omega)},$$

for any $\phi \in C_0^\infty(\Omega)$. Therefore the measure Lu defines a continuous linear functional on $W_0^{2,q'}(\Omega)$. Consequently λ is the sum of an integrable function and a measure in $W^{-2,q}(\Omega)$.

(ii) Conversely, we first assume that λ is a positive measure. By Proposition 3.17 there exists an increasing sequence of positive measures λ_j belonging to $W^{-2,q}$ converging to λ in the weak sense of measures. By Theorem 3.10 there exists u_j solution to

$$\begin{aligned} Lu_j + |u_j|^{q-1}u_j &= \lambda_j \quad \text{in } \Omega, \\ u_j &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.66}$$

Moreover u_j is nonnegative and $u_j \geq u_{j-1}$ for any $j \in \mathbb{N}_*$. For any $\zeta \in C_c^{1,L}(\overline{\Omega})$ there holds

$$\int_\Omega \left(u_jL^*\zeta + u_j^q\zeta\right) dx = \int_\Omega \zeta d\lambda_j. \tag{3.67}$$

Let $u = \lim_{j\to\infty} u_j$. If $\zeta \geq 0$, we have, by the Beppo-Levi Theorem,

$$\int_\Omega (uL^*\zeta + u^q\zeta)\, dx = \int_\Omega \zeta d\lambda. \tag{3.68}$$

Hence $u \in L^1(\Omega) \cap L^q(\Omega; \rho_{\partial\Omega} dx)$ and u is the solution to Problem (3.64). Because λ is bounded we have $u \in L^q(\Omega)$ by Proposition 3.2.

If λ is no longer positive, λ_+ and λ_- do not charge Borel sets with $C_{2,q'}$-capacity zero. Hence there exist two nondecreasing sequences of positive measures belonging to $W^{-2,q}(\Omega)$, $\{\lambda_{j,+}\}$ and $\{\lambda_{j,-}\}$, converging to λ_+ and λ_- respectively. As in the proof of Theorem 3.10 we truncate the nonlinearity by putting $g_k(r) = \text{sign }(r) \min\{k^q, |r|^q\}$ for $k \in \mathbb{N}_*$, and we denote by v_k, (resp. $v_{k,+}$ and $v_{k,-}$) the solutions of

$$\begin{aligned} Lv + g_k(v) &= \nu \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.69}$$

when $\nu = \lambda_{j,+} - \lambda_{j,-}$ (resp. $\nu = \lambda_{j,+}$ and $\nu = \lambda_{j,-}$). By Theorem 3.7, $-v_{k,-} \leq v_k \leq v_{k,+}$, which implies $-g_k(v_{k,-}) \leq g_k(v_k) \leq g_k(v_{k,+})$. When $k \to \infty$, the sequences $\{v_{k,+}\}$ and $\{v_{k,-}\}$ decrease and converge respectively to $u_{j,+}$ and $u_{j,-}$, the solutions of (3.64) with respective right-hand side $\lambda_{j,+}$ and $\lambda_{j,-}$. Moreover

$$-\left(\mathbb{G}_L^\Omega(\lambda_{j,-})\right)^q \leq -g_k(\mathbb{G}_L^\Omega(\lambda_{j,-})) \leq g_k(v_k) \leq g_k\left(\mathbb{G}_L^\Omega(\lambda_{j,+})\right) \leq \left(\mathbb{G}_L^\Omega(\lambda_{j,+})\right)^q. \tag{3.70}$$

Since the left and right-hand side terms are $L^1(\Omega)$-functions, the sequence $\{g_k(v_k)\}$ is uniformly integrable. As in the proof of Theorem 3.10, the sequence $\{v_k\}$ converges in $L^q(\Omega)$ to the solution u_j of (3.66) with right-hand side $\lambda_{j,+} - \lambda_{j,-}$. Furthermore

$$-u_{j,-} \leq u_j \leq u_{j,+}, \quad \text{and} \quad -u_{j,-}^q \leq |u_j|^{q-1} u_j \leq u_{j,+}^q.$$

Because $\{u_{j,+}\}$ and $\{u_{j,-}\}$ are monotone and converge in $L^q(\Omega)$, the sequence $\{u_{j,}\}$ is uniformly integrable in $L^q(\Omega)$ and converges a.e. in Ω. Since $\lambda_{j,+} - \lambda_{j,-}$ converges weakly to λ in the sense of measures, there exists a function $u \in L^q(\Omega)$, solution of (3.64). $\square$

3.4 Removable singularities

3.4.1 Positive solutions

In this section Ω is an arbitrary open set in $\mathbb{R}^n$. Let L_m be a linear differential operator of order m ($m \in \mathbb{N}_*$), defined by

$$L_m u = \sum_{0 \leq |\alpha| \leq m} D^\alpha(a_\alpha u), \tag{3.71}$$

where

$$a_\alpha \in L^\infty_{loc}(\Omega), \quad \forall \alpha \in \mathbb{N}^n,\ |\alpha| \leq m. \tag{3.72}$$

Definition 3.22 Let $G \subset \Omega$ be open, $u \in L^1_{loc}(G)$ and T a distribution on G. We shall say that u satisfies

$$L_m u = T \quad (\text{resp. } L_m u \leq T) \quad \text{in } \mathcal{D}'(G), \tag{3.73}$$

or, equivalently, that u is a distribution solution (resp. subsolution) of (3.73), if

$$\begin{aligned} &\int_G u L_m^* \zeta dx = \langle T, \zeta \rangle \quad (\text{resp. } \int_G u L_m^* \zeta dx \leq \langle T, \zeta \rangle), \\ &\qquad\qquad\qquad\qquad \forall \zeta \in C_c^\infty(G) \quad (\text{resp. } \quad \forall \zeta \in C_c^\infty(G)\,,\, \zeta \geq 0), \end{aligned} \tag{3.74}$$

where $\langle .,. \rangle$ denote the duality pairing between $\mathcal{D}'(G)$ and $\mathcal{D}(G)$, and L_m^* the formal adjoint of L_m defined by

$$L_m^* \zeta = \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} a_\alpha D^\alpha \zeta. \tag{3.75}$$

The following result is due to Baras and Pierre [9].

Theorem 3.23 *Let $m \in \mathbb{N}_*$, $q > 1$, F be a relatively closed subset of G, λ a Radon measure which does not charge sets with $C_{m,q'}$-capacity zero and g a continuous real valued function which satisfies*

$$\liminf_{r \to \infty} g(r)/r^q > 0. \tag{3.76}$$

Let $u \in L^1_{loc}(\Omega \setminus F)$, such that $u \geq 0$ and $g(u) \in L^1_{loc}(\Omega \setminus F)$, be a solution of

$$L_m u + g(u) \leq \lambda \quad \text{in } \mathcal{D}'(\Omega \setminus F). \tag{3.77}$$

If $C_{m,q'}(F) = 0$, then $u \in L^1_{loc}(\Omega)$, $g(u) \in L^1_{loc}(\Omega)$ and there holds

$$L_m u + g(u) \leq \lambda \quad \text{in } \mathcal{D}'(\Omega). \tag{3.78}$$

Proof. Step 1 Let $\zeta \in C_c^\infty(\Omega)$, and $K = \text{supp}(\zeta)$. Since $K \cap F$ is a compact subset of Ω with $C_{m,q'}$-capacity zero, it follows by Proposition 3.14 that there exists a sequence $\{\phi_n\} \subset C_c^\infty(\Omega)$ such that $0 \leq \phi_n \leq 1$, $\phi_n \equiv 1$ in a neighborhood of $K \cap F$ and $\phi_n \to 0$ as $n \to \infty$, in $W^{m,q'}(\Omega)$ and $C_{m,q'}$-quasi everywhere. Therefore, $\zeta_n = (1 - \phi_n)\zeta$ satisfies :

(i) $\zeta_n \in C_c^\infty(\Omega \setminus F)$,

(ii) $0 \leq \zeta_n \leq 1$,

(iii) $\zeta_n \to \zeta$ in $W^{m,q'}(\Omega)$ and $C_{m,q'}$-quasi everywhere as $n \to \infty$, and the sequence $\{\zeta_n\}$ is increasing.

Step 2 We claim that $g(u) \in L^1_{loc}(\Omega)$. We take $\zeta \in C_c^\infty(\Omega)$, $\zeta \geq 0$ and $\{\zeta_n\}$ be defined by the procedure in Step 1. Let $p \in \mathbb{N}$, $p \geq mq'$. Since $\zeta_n^p \in C_c^\infty(\Omega \setminus F)$, (3.77) implies

$$\int_\Omega \left(u L_m^*(\zeta_n^p) + g(u)\zeta_n^p\right) dx \leq \int_\Omega \zeta_n^p d\lambda. \tag{3.79}$$

Because $\zeta_n^p \leq \zeta$, there holds

$$\int_\Omega g(u)\zeta_n^p dx \leq \int_\Omega \zeta d\,|\lambda| + \int_\Omega u\,|L_m^*(\zeta_n^p)|\, dx. \tag{3.80}$$

Since the a_α are locally bounded,

$$|L_m^*(\zeta_n^p)| \leq C \sum_{0\leq|\alpha|\leq m} |D^\alpha(\zeta_n^p)|.$$

The zero order term is estimated by

$$\int_\Omega u\zeta_n^p dx \leq \left(\int_\Omega u^q\zeta_n^p dx\right)^{1/q} \left(\int_\Omega \zeta_n^p dx\right)^{1/p'} \leq \left(\int_\Omega u^q\zeta_n^p dx\right)^{1/q'} \|\zeta_n\|_{W^{m,q'}(\Omega)}. \tag{3.81}$$

If $|\alpha| \geq 1$,

$$D^\alpha(\zeta_n^p) = \sum_{j=1}^{|\alpha|} c_j\zeta_n^{p-j} \sum_{\substack{|\beta_i| \geq 1 \\ \beta_1+...+\beta_j=\alpha}} c_{\beta_1,...,\beta_j} D^{\beta_1}\zeta_n ... D^{\beta_j}\zeta_n,$$

where the c_j and $c_{\beta_1,...,\beta_j}$ are positive constants depending on the indices. Thus we are led to estimate a finite sum involving terms of the form

$$A = \int_\Omega u\zeta_n^{p-j} \left|D^{\beta_1}\zeta_n ... D^{\beta_j}\zeta_n\right| dx.$$

By Hölder's inequality

$$A \leq \left(\int_\Omega u^q\zeta_n^p dx\right)^{1/q} \left(\int_\Omega \zeta_n^{p-jq'} \left|D^{\beta_1}\zeta_n ... D^{\beta_j}\zeta_n\right|^{q'} dx\right)^{1/q'}.$$

Because $p \geq mq' \geq jq'$, it follows $0 \leq \zeta_n^{p-jq'} \leq 1$. By applying again Hölder's inequality, and using the fact that $|\beta_1| + ... + |\beta_j| = |\alpha|$, it follows

$$A \leq \left(\int_\Omega u^q\zeta_n^p dx\right)^{1/q} \prod_{i=1}^{j} \left(\int_\Omega \left|D^{\beta_i}\zeta_n\right|^{q'|\alpha|/|\beta_i|} dx\right)^{|\beta_i|/|\alpha|q'}.$$

By the Gagliardo-Nirenberg inequality, there holds

$$\left|D^{\beta_i}\zeta_n\right|^{q'|\alpha|/|\beta_i|} \leq C\|\zeta_n\|^{|\beta_i|/|\alpha|}_{W^{|\alpha|,q'}(\Omega)} \leq C\|\zeta_n\|^{|\beta_i|/|\alpha|}_{W^{m,q'}(\Omega)}.$$

Therefore

$$A \leq C\left(\int_\Omega u^q\zeta_n^p dx\right)^{1/q} \|\zeta_n\|_{W^{m,q'}(\Omega)}, \tag{3.82}$$

from which derives

$$\int_\Omega g(u)\zeta_n^p dx \leq C_1 + C_2\left(\int_\Omega u^q\zeta_n^p dx\right)^{1/q} \|\zeta_n\|_{W^{m,q'}(\Omega)}. \tag{3.83}$$

By assumption, there exist two positive constants a and b such that

$$g(r) \geq ar^q - b, \quad \forall r \geq 0.$$

Consequently, up to changing the constants C_i,

$$\int_\Omega (g(u)+b)\zeta_n^p dx \leq C_1 + C_2 \left(\int_\Omega (g(u)+b)\zeta_n^p dx \right)^{1/q} \|\zeta_n\|_{W^{m,q'}(\Omega)}. \tag{3.84}$$

Finally, the left-hand side integral remains bounded independently of n and we conclude by Fatou's lemma that $(g(u)+b)\zeta^p \in L^1(\Omega)$. Since ζ is arbitrary, we find $g(u) \in L^1_{loc}(\Omega)$. The growth estimate on g implies also $u \in L^q_{loc}(\Omega)$.

Step 3 We claim that (3.78) holds. Let $\zeta \in C_c^\infty(\Omega)$, $\zeta \geq 0$. By constructing the same functions ζ_n as above, we have

$$\int_\Omega (u L_m^* \zeta_n + g(u)\zeta_n)\, dx \leq \int_\Omega \zeta_n d\lambda. \tag{3.85}$$

Since $|\lambda|$ does not charge sets with $C_{m,q'}$-capacity zero and $\zeta_n \to \zeta$, $C_{m,q'}$-quasi everywhere in Ω, this convergence holds also $|\lambda|$-a.e. in Ω. By the Lebesgue Theorem

$$\lim_{n\to\infty} \int_\Omega \zeta_n d\lambda = \int_\Omega \zeta d\lambda.$$

Because $g(u)$ is locally integrable in Ω,

$$\lim_{n\to\infty} \int_\Omega g(u)\zeta_n dx = \int_\Omega g(u)\zeta dx,$$

and finally, the convergence of $\{\zeta_n\}$ to ζ in $W^{m,q'}(\Omega)$ implies the convergence of $\{L_m^*\zeta_n\}$ to $L_m^*\zeta$ in $L^{q'}(\Omega)$. Passing to the limit in (3.85) yields to (3.78). $\square$

Remark. Contrary to the case of semilinear elliptic equations with an absorbing nonlinearity, which will be studied in next section, the removability of F does not imply that the function u is regular in whole Ω : the singularity is just not seen at the distributions level.

3.4.2 Semilinear elliptic equations with absorption

The first result of unconditional removability of isolated sets for semilinear elliptic equations with absorption term is due to Brezis and Véron [23]. It deals with equation

$$-\Delta u + g(u) = 0, \tag{3.86}$$

in $\Omega \setminus \{0\}$, where Ω is an open subset of $\mathbb{R}^n$ ($n \geq 3$) containing 0 and g a continuous function. They proved the following.

Theorem 3.24 *Suppose g satisfies*

$$\liminf_{r\to\infty} g(r)/r^{n/(n-2)} > 0 \quad \textit{and} \quad \limsup_{r\to-\infty} g(r)/|r|^{n/(n-2)} < 0. \tag{3.87}$$

If $u \in L^\infty_{loc}(\Omega \setminus \{0\})$ satisfies (3.86) in the sense of distributions in $\Omega \setminus \{0\}$, there exists a function $\tilde{u} \in C^1(\Omega) \cap W^{2,p}_{loc}(\Omega)$ for any $1 \leq p < \infty$, which coincides with u a.e. in Ω, and is a solution of (3.86) in whole Ω.

The proof of this result is settled upon a particular case of a general *a priori* estimate discovered by Keller [53] and Osserman [83] separately. In this particular case, and in assuming that $B_R(0) \subset \Omega$, it reads

$$|u(x)| \leq A\,|x|^{2-n} + B, \quad \forall x \in B_{R/2}(0) \setminus \{0\}, \tag{3.88}$$

for some positive constants A and B. From this estimate is derived the local integrability of u in Ω and then of $g(u)$. Finally, it leads to the fact that Equation (3.86) holds in the sense of distributions in Ω. The conclusion follows by the maximum principle (which implies the boundedness of u near 0), and the elliptic equations regularity theory. Later on, this result was extended by Véron [102] as follows :

Theorem 3.25 *Let $\Sigma \subset \Omega$ be a complete and compact d-dimensional submanifold of class C^2 $(1 \leq d < n-2)$ and g is a continuous real valued function such that*

$$\liminf_{r\to\infty} g(r)/r^{(n-d)/(n-2-d)} > 0 \quad \text{and} \quad \limsup_{r\to-\infty} g(r)/\,|r|^{(n-d/(n-2-d)} < 0. \tag{3.89}$$

If $u \in L^\infty_{loc}(\Omega \setminus \Sigma)$ satisfies (3.86) in the sense of distributions in $\Omega \setminus \Sigma$, there exists a function $\tilde{u} \in C^1(\Omega) \cap W^{2,p}_{loc}(\Omega)$ for any $1 \leq p < \infty$, which coincides with u a.e. in Ω and is a solution of (3.86) in whole Ω.

Although more technical, the idea of the proof is similar to the one of Theorem 3.24, except that the *a priori* estimate (3.88) is replaced by

$$|u(x)| \leq A\,(\operatorname{dist}(x,\Sigma))^{2-n-d} + B, \quad \forall x \in G \setminus \Sigma, \tag{3.90}$$

where G is open and bounded and $\Sigma \subset G \subset \overline{G} \subset \Omega$. The method developed by Baras and Pierre [9] is settled upon integral identity, without using pointwise *a priori* estimates as the previous authors do.

Theorem 3.26 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with a C^2 boundary, L an elliptic operator defined by 2.1 satisfying condition (H) and $q > 1$. If F is a compact subset of Ω, any solution $u \in L^q_{loc}(\Omega \setminus K)$ of*

$$Lu + |u|^{q-1}u = 0, \tag{3.91}$$

in $\Omega \setminus K$, belongs to $L^q_{loc}(\Omega)$ and satisfies (3.91) in whole Ω, if and only if $C_{2,q'}(K) = 0$. If this holds, $u \in W^{2,p}_{loc}(\Omega)$ for any $1 \leq p < \infty$, and (3.91) is satisfied a.e. in Ω.

Proof. (i) Let us assume that $C_{2,q'}(K) > 0$. By (3.54), there exists a positive Radon measure λ concentrated on K such that

$$\int_\Omega |G_2 * \mu|^q dx < \infty.$$

This means that $\lambda \in W^{-2,q}(\Omega)$. By Theorem 3.20, Problem (3.64) admits a solution in Ω.

(ii) Conversely we assume that $C_{2,q'}(K)=0$. By Theorem 2.4, for any $\zeta\in C_c^{1,L}(\Omega\setminus F)$, $\zeta\geq 0$, there holds

$$\int_\Omega \left(|u|\,L^*\zeta+|u|^q\,\zeta\right)dx\leq 0.$$

Therefore $v=|u|$ is a subsolution of (3.91) in the sense of Definition 3.22. Since $C_{2,q'}(K)=0$, we can extend v as a solution of (3.91) in whole Ω, and because K has zero Lebesgue measure, $u\in L^q_{loc}(\Omega)$. Let $\zeta_n=(1-\phi_n)\zeta$ be the functions defined in Theorem 3.23 for an arbitrary $\zeta\in C_c^\infty(\Omega)$ (we do not impose the positivity). Then $\zeta_n\to\zeta$ in $W^{2,q'}(\Omega)$ and $C_{2,q'}$-quasi everywhere. By assumption

$$\int_\Omega \left(uL^*\zeta_n+|u|^{q-1}\,u\zeta_n\right)dx=0.$$

By Lebesgue's theorem, $|u|^{q-1}\,u\zeta_n\to|u|^{q-1}\,u\zeta$ in $L^1(\Omega)$. Moreover $L^*\zeta_n\to L^*\zeta$ in $L^{q'}(\Omega)$. Therefore, by letting $n\to\infty$, it is infered that

$$\int_\Omega \left(uL^*\zeta+|u|^{q-1}\,u\zeta\right)dx=0, \tag{3.92}$$

which proves that (3.91) holds in Ω. Let G be any smooth open domain containing K and such that $\overline{G}\subset\Omega$. For $\beta>0$ small enough we put $G_\beta=\{x\in G:\mathrm{dist}\,(x,\partial G>\beta\}$, and $\Gamma_\beta=\{x\in G:\mathrm{dist}\,(x,\partial G)=\beta\}=\partial G_\beta$. There exists β_0 such that Γ_β is a smooth surface in $\mathbb{R}^n$. Because $u\in L^q(G\setminus\overline{G}_{\beta_0})$, it follows, by Fubini's theorem, that $u\lfloor_{\Gamma_\beta}\in L^q(\Gamma_\beta)$ (endowed with the $(n-1)$-dimensional Hausdorff measure), for almost all $\beta\in[0,\beta_0]$. We fix a β such that this property holds and denote by V the Poisson potential of $u_+\lfloor_{\Gamma_\beta}$ in G_β. By (2.22), for any $\zeta\in C_c^{1,L}(\overline{G}_\beta)$, $\zeta\geq 0$, there holds

$$\int_{G_\beta}\left((u-V)_+L^*\zeta+(u-V)_+\,|u|^{q-1}\,u\zeta\right)dx\leq-\int_{\partial G_\beta}\frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}(u-u_+)_+dS. \tag{3.93}$$

Taking $\zeta=\mathbb{G}_L^{G_\beta}(1)$ implies $(u-V)_+\equiv 0$ in G_β. Thus $u\leq V$ in G_β. Since $V\in L^\infty_{loc}(G_\beta)$, the same property holds with u_+. Since G is arbitrary, $u_+\in L^\infty_{loc}(\Omega)$. In the same way $u_-\in L^\infty_{loc}(\Omega)$. We conclude with the elliptic equations regularity theory that $u\in W^{2,p}_{loc}(\Omega)$. □

Remark. The following extension of Theorem 3.26 is easy to establish : *Let g be a continuous real valued function which satisfies*

$$\liminf_{r\to\infty} g(r)/r^q>0 \quad \textit{and} \quad \limsup_{r\to-\infty} g(r)/\,|r|^q<0, \tag{3.94}$$

for some $q>1$. Let $\lambda\in\mathfrak{M}(\Omega)$ which does not charge sets with $C_{2,q'}$-capacity zero and K a compact subset of Gw with $C_{2,q'}$-capacity zero. Then any function u, locally integrable in $\Omega\setminus K$ and such that $g(u)\in L^1_{loc}(\Omega\setminus K)$, which verifies

$$Lu+g(u)=\lambda, \tag{3.95}$$

in $\mathcal{D}'(\Omega\setminus K)$, can be extended as a solution of the same equation in $\mathcal{D}'(\Omega)$. Furthermore $g(u)\in C(\Omega)$ and $u\in W^{2,p}_{loc}(\Omega)$, for any $1\leq p<\infty$.

3.5 Isolated singularities

The description of the behaviour of solutions of semilinear elliptic equations near an isolated singularity deals with the following question : let u be a solution of

$$Lu + g(u) = 0 \quad \text{in } \Omega \setminus \{0\}, \tag{3.96}$$

where Ω is an open subset of $\mathbb{R}^n$ containing 0, L a elliptic operator under the form (2.2) and g a continuous real-valued function, can one describe the behaviour of $u(x)$ as $x \to 0$? When $L = -\Delta$ and $g = 0$, it is known that u admits an expansion in series of spherical harmonics. For the equation

$$-\Delta u + |u|^{q-1} u = 0 \quad \text{in } \Omega \setminus \{0\}, \tag{3.97}$$

($q > 1$), much work on this subject has been done by Véron in [101]. Notice that if $q \geq n/(n-2)$ Brezis-Véron's result (see Theorem 3.24) applies and the function u is C^2 in whole Ω. When $1 < q < n/(n-2)$ this is no longer the case. For example there exists an explicit radial singular solution of (3.97),

$$x \mapsto u_s(x) = \ell_{q,n} |x|^{-2/(q-1)} \tag{3.98}$$

defined in $\mathbb{R}^n \setminus \{0\}$, where

$$\ell_{q,n} = \left(\left(\frac{2}{q-1} \right) \left(\frac{2q}{q-1} - n \right) \right)^{1/(q-1)}. \tag{3.99}$$

When $1 < q < (n+1)/(n-1)$ there exist separable singular solutions. For expressing them, let (r, σ) be the spherical coordinates in $\mathbb{R}^n$ and $\Delta_{S^{n-1}}$ the Laplace-Beltrami operator on the unit sphere $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. If $1 < q < (n+1)/(n-1)$, one has $\ell_{q,n} > n - 1 = \lambda_1(S^{n-1})$, the first nonzero eigenvalue of $\Delta_{S^{n-1}}$. Therefore, the classical variational analysis applies and there exist non-trivial solutions of

$$-\Delta_{S^{n-1}} \omega - \ell_{q,n} \omega + |\omega|^{q-1} \omega = 0 \quad \text{in } S^{n-1}. \tag{3.100}$$

Hence the function

$$x \mapsto u_\omega(x) = u_\omega(r, \sigma) = r^{-2/(q-1)} \omega(\sigma) \tag{3.101}$$

is a singular solution of (3.97). Notice that u_s is one of these solutions. Furthermore the constants $\ell_{q,n}$ and $-\ell_{q,n}$ are the only solutions of (3.100) which have a constant sign. The following result is proven in [101].

Theorem 3.27 *Let $1 < q < n/(n-2)$ ($q > 1$ if $n = 2$) and u be positive solution of (3.97) in some open set Ω containing 0. Then,*
(i) either

$$\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \ell_{q,n}, \tag{3.102}$$

(ii) or there exists some $c \geq 0$ such that

$$\lim_{x \to 0} |x|^{n-2} u(x) = c, \tag{3.103}$$

if $n \geq 3$, and $|x|^{n-2}$ replaced by $1/\ln(1/|x|)$ in the above formula if $n = 2$. Furthermore u is a solution of

$$-\Delta u + u^q = C_n c \delta_0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.104}$$

for some positive constant C_n depending only on n.

There are several proofs of this result, based either on a sharp use of the radial case and the Harnack inequality, or on a Lyapounov style analysis. If the function u is no longer supposed to have constant sign, it is proven in [101] that the above dichotomy still holds provided $(n+1)/(n-1) \leq q < n/(n-2)$. However (i) has to be replaced by

(i') either

$$\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \ell \in \{\ell_{q,n}, -\ell_{q,n}\}, \tag{3.105}$$

and (ii) by

(ii') or there exists some real number c such that

$$\lim_{x \to 0} |x|^{n-2} u(x) = c, \tag{3.106}$$

(if $n \geq 3$, with the classical modification if $n = 2$). Moreover u is a solution of

$$-\Delta u + |u|^{q-1} u = C_n c \delta_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{3.107}$$

Actually, the Lyapounov analysis leads easily to a more general result [27].

Theorem 3.28 *Let $1 < q < n/(n-2)$ and u be solution of (3.97) in some open set Ω containing 0. Then there exists a compact and connected subset $\mathcal{E}$ of the set of solutions of (3.100) such that*

$$\lim_{r \to 0} \operatorname{dist}_{C^2(S^{n-1})}(r^{2/(q-1)} u(r,.), \mathcal{E}) = 0, \tag{3.108}$$

where $\operatorname{dist}_{C^2(S^{n-1})}$ denotes the distance associated to the $C^2(S^{n-1})$-norm.

This result leaves open two difficult questions :

1- Does it exist a particular element $\omega \in \mathcal{E}$ such that

$$\lim_{r \to 0} \left\| r^{2/(q-1)} u(r,.) - \omega \right\|_{C^2(S^{n-1})} = 0\,? \tag{3.109}$$

2- What is the precise behaviour of u when $\mathcal{E} = \{0\}$?

Besides the results above mentioned proven in [101], the two questions have been thoroughly answered in [27] in the two-dimensional case.

Theorem 3.29 *Assume $n = 2$, $q > 1$ and u is solution of (3.97) in $\Omega \setminus \{0\}$. Then there exists a 2π-periodic function ω, solution of*

$$-\frac{d^2\omega}{d\sigma^2} - \left(\frac{2}{q-1}\right)^2 \omega + |\omega|^{q-1}\,\omega = 0 \tag{3.110}$$

such that (3.109) holds on S^1.

Theorem 3.30 *Under the assumption of Theorem 3.29, if $\omega = 0$, let k_0 be the largest integer smaller than $2/(q-1)$. Then*
(i) either there exist an integer $k \in [1, k_0]$ and two constants $A \neq 0$ and $\phi \in S^1$ such that

$$\lim_{r \to 0} r^k u(r,\sigma) = A \sin(k\sigma + \phi), \tag{3.111}$$

in the $C^2(S^{n-1})$-topology,
(ii) or there is a nonzero c such that

$$\lim_{r \to 0} u(r,\sigma)/\ln(1/r) = c, \tag{3.112}$$

in the $C^2(S^{n-1})$-topology,
(iii) or u can be extended as a C^2 solution of (3.97) in whole Ω.
In cases (ii) and (iii), u is a solution of (3.107) in $\mathcal{D}'(\Omega)$.

The proofs are extremely technical and use, in a fundamental manner, the Sturmian argument about the oscillations of solutions of 2 dimensional elliptic equations jointly with the Jordan curve separation Theorem.

Many of the above results can be extended in a standard way to elliptic equations of the type

$$Lu + |u|^{q-1}\,u = 0, \tag{3.113}$$

where L is the elliptic operator defined by (2.1) subject to condition (H), and assuming $a_{ij}(x) = a_{ji}(x)$, an assumption which is not a real restriction. If we fix a linear change of variable in $\mathbb{R}^n$, $y = y(x)$, and write $u(x) = \tilde{u}(y)$, then

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l} b_{li} b_{kj} \frac{\partial^2 \tilde{u}}{\partial y_l \partial y_k},$$

where

$$b_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}.$$

Then

$$\sum_{i,j} a_{ij}(0) \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l} \frac{\partial^2 \tilde{u}}{\partial y_l \partial y_k} \sum_{i,j} a_{ij}(0) b_{li} b_{kj}.$$

Since the matrix $(a_{ij}(0))$ is symmetric, the $b_{\alpha\beta}$ can be chosen such that

$$\sum_{i,j} a_{ij}(0) b_{li} b_{kj} = \delta_{kl}.$$

With this transformation most of the above results can be restated with the variable y replacing x. For example Theorem 3.27 transforms into

Theorem 3.31 *Let* $1 < q < n/(n-2)$ *and* u *be positive solution of (3.113) in some open set* Ω *containing* 0. *Then,*
(i) either

$$\lim_{y \to 0} |y|^{2/(q-1)}\, \tilde{u}(y) = \ell_{q,n}, \tag{3.114}$$

(ii) or there exists some $c \geq 0$ *such that*

$$\lim_{y \to 0} |y|^{n-2}\, \tilde{u}(y) = c, \tag{3.115}$$

in which case u *is a solution of*

$$Lu + u^q = C_{n,L} c \delta_0 \quad \textit{in } \mathcal{D}'(\Omega), \tag{3.116}$$

for some positive constant $C_{n,L}$ *depending only on* n *and* L.

The description given by (3.105) of isolated singularities in the case of signed solutions of (3.113) holds in the new unknown $\tilde{u}$ and variable y, provided $(n+1)/(n-1) < q < n/(n-2)$, and similarly the method which gives (3.108) applies without restriction. However the sharp analysis of the the limit case $q = (n+1)/(n-1)$ when the limit set is reduced to the zero function cannot be covered by this rough analysis. Moreover, the extension of the results given in [27] (even in the non-critical cases where $2/(q-1)$ is not an integer) has not yet been done.

3.6 The exponential and 2-dimensional cases

3.6.1 Unconditional solvability

As we have seen it above, the Bénilan-Brezis weak-singularity assumption [11] is meaningless in the 2-dimensional case for solving semilinear elliptic equations with bounded measures : the $(n, 0)$- weak-singularity assumption imposes $n \geq 3$ in Definition 3.6. If $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, L an elliptic operator, $g \in C(\Omega \times \mathbb{R})$ is an absorbing nonlinearity and $\lambda \in \mathfrak{M}^b(\Omega)$, a specific approach, developped by Vazquez [94], is needed, for solving

$$\begin{aligned} Lu + g(x, u) &= \lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.117}$$

Definition 3.32 Let $\tilde{g} \in C([0,\infty))$, $\tilde{g} \geq 0$. We denote by

$$a_+(\tilde{g}) := \inf\left\{a \geq 0 : \int_0^\infty \tilde{g}(s)e^{-as}ds < \infty\right\}, \tag{3.118}$$

the *exponential order of growth of* $\tilde{g}$ *at infinity.*

If $g^* \in C((-\infty,0])$, $g^* \leq 0$, the *exponential order of growth of* g^* *at minus infinity* is by definition the opposite of the exponential order of growth at infinity of the function $r \mapsto -g^*(-r)$, thus

$$a_-(g^*) := \sup\left\{a \leq 0 : \int_{-\infty}^0 g^*(s)e^{as}ds > -\infty\right\}. \tag{3.119}$$

Those two quantities may be zero (for example if $\tilde{g}$ is a power), finite and nonzero (if $\tilde{g}$ is an exponential) or infinite (if $\tilde{g}$ is a super-exponential).

Definition 3.33 A real valued function $g \in C(\Omega \times \mathbb{R})$ satisfies the 2-*dimensional weak-singularity assumption*, if there exists $r_0 \geq 0$ such that

$$rg(x,r) \geq 0, \quad \forall (x,r) \in \Omega \times (-\infty,-r_0] \cup [r_0,\infty), \tag{3.120}$$

and two nondecreasing functions $\tilde{g}_1 \in C([0,\infty))$, $\tilde{g}_1 \geq 0$, with zero exponential order of growth at infinity, and $\tilde{g}_2 \in C((-\infty,0])$, $\tilde{g}_2 \leq 0$, with zero exponential order of growth at minus infinity such that

$$g(x,r) \leq \tilde{g}_1(r), \quad \forall (x,r) \in \Omega \times \mathbb{R}_+, \tag{3.121}$$

and

$$\tilde{g}_2(r) \leq g(x,r), \quad \forall (x,r) \in \Omega \times \mathbb{R}_-. \tag{3.122}$$

Notice that the zero exponential of growth assumptions can be written under the form

$$\int_0^\infty (\tilde{g}_1(s) - \tilde{g}_2(-s))\, e^{-as}ds < \infty, \quad \forall a > 0. \tag{3.123}$$

Theorem 3.34 *Let $\Omega \subset \mathbb{R}^2$ be a C^2 bounded domain and $g \in C(\Omega \times \mathbb{R})$ satisfy the 2-dimensional weak-singularity assumption. For any $\lambda \in \mathfrak{M}_b(\Omega)$ Problem (3.117) admits a solution. Furthermore, δ is invariant if we replace g by ℓg, for any $\ell > 0$.*

One of the tool of the proof is John-Nirenberg's theorem [47, Th. 7.21].

Theorem 3.35 *Let G be a convex open domain in $\mathbb{R}^n$ and $v \in W^{1,1}(G)$. Assume that there exists $K > 0$ such that*

$$\int_{G\cap B_r(a)} |\nabla v|\, dx \leq Kr^{n-1}, \quad \forall a \in G,\ \forall r > 0. \tag{3.124}$$

Then there exist two positive constants C and μ_0, depending only on n, such that

$$\int_G \exp\left(\frac{\mu}{K}|v - v_G|\right) dx \leq C\,(\mathrm{diam}(G))^n\,, \tag{3.125}$$

where $\mu = \mu_0\,|G|\,(\mathrm{diam}(G))^{-n}$, and $v_G = \dfrac{1}{|G|}\displaystyle\int_G vdx$.

Notice that for any bounded domain $G \subset \mathbb{R}^n$, $\operatorname{diam}(G) = \operatorname{diam}(\operatorname{conv} G)$. Then the following consequence of Theorem 3.35 is valid.

Corollary 3.36 *Let G be a bounded open domain in $\mathbb{R}^n$ and $v \in W_0^{1,1}(G)$. Assume that there exists $K > 0$ such that (3.124) holds. Then there exist two positive constants C and μ_0, depending only on n, such that (3.125) holds with $\mu = \mu_0 \left|\operatorname{conv} G\right| (\operatorname{diam}(G))^{-n}$ and v_G replaced by $v_{\operatorname{conv} G} = \dfrac{1}{|\operatorname{conv} G|}\displaystyle\int_G v dx$.*

Proof of Theorem 3.34. *Step 1* Approximation. First we multiply λ by the characteristic function χ_{Ω_n} of $\Omega_n = \{x \in \Omega : \rho_{\partial\Omega}(x) > 1/n\}$, and we regularize $\chi_{\Omega_n}\lambda$ by convolution with positive smooth functions with compact support and total mass 1. By the property of convolution can replace λ_+ and λ_- by λ_{n+} and $\lambda_{n-} \in C_c^\infty(\Omega)$, and they satisfy,

$$\|\lambda_{n+}\|_{L^1(\Omega)} \leq \|\lambda_+\|_{\mathfrak{M}^b(\Omega)},$$

and

$$\|\lambda_{n-}\|_{L^1(\Omega)} \leq \|\lambda_-\|_{\mathfrak{M}^b(\Omega)}.$$

Let u_n be the solution of

$$\begin{aligned} Lu_n + g(x, u_n) &= \lambda_n \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.126}$$

Such a problem admits solutions (see Steps 1-3 of the proof of Theorem 3.7). The following two estimates hold

$$\|u_n\|_{L^1(\Omega)} + \|\rho_{\partial\Omega} g(., u_n)\|_{L^1(\Omega)} \leq \Theta \int_\Omega \rho_{\partial\Omega} dx + C_1 \|\lambda_n\|_{L^1(\Omega)} \leq C_2, \tag{3.127}$$

where $-\Theta \leq \min\{\operatorname{sign}(r) g(x,r) : (x,r) \in \Omega \times \mathbb{R}\}$ is nonpositive, and

$$\|\nabla u_n\|_{M^2(\Omega)} \leq C_4(\Theta + \|\lambda_n\|_{L^1(\Omega)}) \leq C_5. \tag{3.128}$$

Notice that (3.128), which replaces (3.25), follows from (3.10). As in the proof of Theorem 3.7 there exist a subsequence $\{u_{n_k}\}$ and a function $u \in W_0^{1,q}(\Omega)$, for any $1 \leq q < 2$, such that $u_{n_k} \to u$ in $L^1(\Omega)$ and a.e. in Ω.

Step 2 Convergence. Because (3.128) holds,

$$\int_{\Omega \cap B_r(a)} |\nabla u_n| \, dx \leq C_5 |\Omega \cap B_r(a)|^{1/2} \leq C_5 \sqrt{\pi} r, \quad \forall r > 0, \ a \in \Omega, \tag{3.129}$$

and Corollary 3.36 implies

$$\int_\Omega \exp(\mu \left|u_n\right| / C_5 \sqrt{\pi}) dx \leq C_6 \left|\Omega\right| \exp(\mu \left|u_{n \operatorname{conv} \Omega}\right| / C_5 \sqrt{\pi}) \leq C_7, \tag{3.130}$$

since $\|u_n\|_{L^1(\Omega)}$ is uniformly bounded. If we set

$$\theta_n(s) = \int_{\{x \in \Omega : |u_n(x)| > s\}} dx \quad \text{and} \quad \beta = \frac{\mu}{C_5 \sqrt{\pi}},$$

then

$$0 \leq \theta_n(s) \leq C_7 e^{-\beta s}, \quad \forall s \geq 0. \tag{3.131}$$

Let ω be any Borel subset of Ω. As in Theorem 3.7-Step 3, for any $R > 0$, we have

$$\begin{aligned}
\int_\omega |g(x, u_n| \, dx &\leq \int_\omega (\tilde{g}_1(|u_n|) - \tilde{g}_2(-|u_n|)) \, dx, \\
&\leq (\tilde{g}_1(R) - \tilde{g}_2(-R)) \, |\omega| - \int_R^\infty (\tilde{g}_1(s) - \tilde{g}_2(-s)) d\theta_n(s).
\end{aligned}$$

Therefore, as in the proof of Theorem 3.7,

$$\begin{aligned}
\int_R^\infty (\tilde{g}_1(s) - \tilde{g}_2(-s)) d\theta_n(s) &= (\tilde{g}_1(R) - \tilde{g}_2(-R))\theta_n(R) + \int_R^\infty \theta_n(s) d(\tilde{g}_1(s) - \tilde{g}_2(-s)), \\
&\leq (\tilde{g}_1(R) - \tilde{g}_2(-R))\theta_n(R) + C_7 \int_R^\infty e^{-\beta s} d(\tilde{g}_1(s) - \tilde{g}_2(-s)), \\
&\leq \frac{C_7}{\beta} \int_R^\infty (\tilde{g}_1(s) - \tilde{g}_2(-s)) e^{-\beta s} ds.
\end{aligned}$$

Let $\epsilon > 0$ arbitrary. By (3.123) there exists $R > 0$ such that

$$\frac{C_7}{\beta} \int_R^\infty (\tilde{g}_1(s) - \tilde{g}_2(-s)) e^{-\beta s} ds \leq \epsilon/2.$$

Now

$$|\omega| \leq \epsilon/2(1 + \tilde{g}_1(R) - \tilde{g}_2(-R)) \Longrightarrow \int_\omega |g(x, u_n)| \, dx \leq \epsilon.$$

We conclude by the Vitali Theorem that $g(., u_{n_k}) \to g(., u)$ in $L^1(\Omega)$, and we end the proof as for Theorem 3.7. □

If $g(x, r) = e^{ar}$ for some $a > 0$, the previous result does not apply for any bounded measure λ. However, if the constant C_5 is small enough, which means that Θ and $\|\lambda\|_{\mathfrak{M}^b(\Omega)}$ are, accordingly, small, the uniform integrability may hold. The proof of the following variant is parallel to the one of Theorem 3.34.

Theorem 3.37 *Let $\Omega \subset \mathbb{R}^2$ be a C^2 bounded domain and $g \in C(\Omega \times \mathbb{R})$ with finite exponential orders of growth at plus and minus infinity. Then there exists $\delta > 0$ such that for any $\lambda \in \mathfrak{M}^b(\Omega)$, if $\|\lambda\|_{\mathfrak{M}^b(\Omega)} \leq \delta$, Problem (3.117) admits a solution.*

The monotonicity and uniform integrability arguments imply also the following stability result.

Corollary 3.38 *Let $\Omega \subset \mathbb{R}^2$ be a C^2 bounded domain and $g \in C(\Omega \times \mathbb{R})$ satisfy the 2-dimensional weak-singularity assumption. Assume also that $r \mapsto g(x, r)$ is nondecreasing for any $x \in \Omega$. Then, for any $\lambda \in \mathfrak{M}_b(\Omega)$, the solution u of Problem (3.117) is unique and the mapping $\lambda \mapsto u$ is nondecreasing. Furthermore, if $\{\lambda_m\}$ is a sequence of bounded measures in Ω which converges in the sense of measures to λ, the corresponding solutions u_m to problem (3.117) converge to u in $L^1(\Omega)$.*

3.6.2 Subcritical measures

For simplicity we shall consider only nondecreasing absorption nonlinearities $g \in C(\mathbb{R})$ in the problem

$$\begin{aligned} -\Delta u + g(u) &= \lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.132}$$

where Ω is a smooth bounded domain of the plane, and $\lambda \in \mathfrak{M}^b(\Omega)$.

Definition 3.39 Let λ be a bounded measure in Ω, with Lebesgue decomposition $\lambda = \lambda^* + \lambda_s + \sum_{j\in J} c_j \delta_{x_j}$ where λ^* is the absolutely continuous part with respect to the 2-dimensional Hausdorff measure, λ_s the singular non-atomic part and $\{(c_j, x_j)\}_{j\in J}$ the set, at most countable, of atoms. Let g be a continuous nondecreasing real valued function. We say that λ is *subcritical with respect to g* if

$$\frac{4\pi}{a_-(g)} \leq c_j \leq \frac{4\pi}{a_+(g)}, \quad \forall j \in J. \tag{3.133}$$

The following result is due to Vazquez [94].

Theorem 3.40 *Let $\lambda \in \mathfrak{M}^b(\Omega)$. Problem (3.132) admits a solution if and only if λ is subcritical with respect to g.*

The local version of the necessary condition is the following.

Proposition 3.41 *Assume g has positive and finite exponential order of growth at infinity, $a_+(g)$. Let $R > 0$ and $\nu \in \mathfrak{M}^b(B_R(0))$ with no atom. If $c > 4\pi/a_+(g)$ there exists no function $u \in L^1(B_R(0))$ such that $g(u) \in L^1(B_R(0))$ and*

$$\int_{B_R(0)} (-u\Delta\zeta + g(u)\zeta)\, dx = c\zeta(0) + \int_{B_R(0)} \zeta d\nu, \quad \forall \zeta \in C_c^\infty(B_R(0)). \tag{3.134}$$

The next result is a particular case of a remarkable relaxation phenomenon which occurs above the critical level $4\pi/a_+(g)$. We denote by B_R the ball of center 0 and radius R and by $B_R^* = B_{R\setminus\{0\}}$.

Lemma 3.42 *Let g be a continuous nondecreasing function with positive and finite exponential order of growth at infinity $a_+(g)$ and, for $n \in \mathbb{N}_*$, $g_n(r) = \min\{g(r), g(n)\}$. Let $R > 0$, $c > c_+(g) = 4\pi/a_+(g)$ and b be three constants, and υ_n the solution of*

$$\begin{aligned} -\Delta \upsilon_n + g_n(\upsilon_n) &= c\delta_0 \quad \textit{in } \mathcal{D}'(B_R), \\ \upsilon_n &= b \quad \textit{on } \partial B_R. \end{aligned} \tag{3.135}$$

When $n \to \infty$, $\{\upsilon_n\}$ decreases and converges, locally uniformly in B_R^*, to the solution $\upsilon_{c_+(g)}$ of

$$\begin{aligned} -\Delta \upsilon_{c_+(g)} + g(\upsilon_{c_+(g)}) &= c_+(g)\delta_0 \quad \textit{in } \mathcal{D}'(B_R), \\ \upsilon_{c_+(g)} &= b \quad \textit{on } \partial B_R. \end{aligned} \tag{3.136}$$

Proof. Since $a_+(g_n) = 0$, we know by Theorem 3.34, that for any $c > 0$, there exists a unique solution υ_n to (3.135), which is therefore a radially symmetric function. Because g_n is increasing, the sequence $\{\upsilon_n\}$ is nonincreasing.

Step 1 Existence of a solution to problem (3.136) in the case $c < c_+(g)$. By comparing υ_n with the solution $\Psi = \Psi_c$ of

$$\begin{aligned} -\Delta\Psi &= c\delta_0 + |g(0)| && \text{in } \mathcal{D}'(B_R), \\ \Psi &= |b| && \text{on } \partial B_R, \end{aligned} \tag{3.137}$$

there holds $\Psi \geq \max\{0, \upsilon_n\}$. But Ψ has the explicit form

$$\Psi(x) = \frac{c}{2\pi}\ln(1/|x|) + K. \tag{3.138}$$

for some constant K. The function υ_n is bounded from below by the solution Φ of

$$\begin{aligned} -\Delta\Phi + g(\Phi) &= 0 && \text{in } \mathcal{D}'(B_R), \\ \Phi &= b && \text{on } \partial B_R, \end{aligned} \tag{3.139}$$

and Φ is a bounded function. Therefore, for n large enough,

$$g(\Phi) \leq g_n(\upsilon_n) \leq g(\upsilon_n) \leq g(\Psi) = g\left(\frac{c}{2\pi}\ln(1/|x|) + K\right).$$

But

$$\int_{B_R} g\left(\frac{c}{2\pi}\ln(1/|x|) + K\right) dx \leq \int_{B_R} g\left(\frac{c}{2\pi}\ln(k/|x|)\right) dx = \frac{2k\pi}{c}\int_\rho^\infty g(s)e^{-4\pi s/c}ds,$$

for some $k > 0$, $\rho > 0$. This last integral is finite because $4\pi/c > a_+(g)$. We conclude with Lebesgue's theorem that υ_n converges to the solution υ_c to (3.136).

Step 2 Existence of a solution to problem (3.136) in the case $c = c_+(g)$. Let $\{c_n\}$ be a positive increasing sequence converging to $c_+(g)$. Then the sequence $\{\upsilon_{c_n}\}$ is increasing. Since $\Phi \leq \upsilon_{c_n} \leq \Psi_{c_+}$ (given by (3.129) and (3.130)), the limit υ^* of the υ_{c_n} is attained in the $L^1(B_R)$-norm, and

$$\Phi \leq \upsilon^* \leq \Psi_{c_+}.$$

The sequence $\{g(\upsilon_{c_n})\}$ is increasing and converges pointwise to $g(\upsilon^*)$. Let $\eta_1 \in C_c^2(\overline{B_R})$ be the solution of

$$\begin{aligned} -\Delta\eta_1 &= 1 && \text{in } B_R, \\ \eta_1 &= b && \text{on } \partial B_R. \end{aligned} \tag{3.140}$$

Hence $\eta_1 \geq 0$ and

$$\int_{B_R} (-\upsilon_{c_n}\Delta\eta_1 + g(\upsilon_{c_n})\eta_1)\, dx = c_n\eta_1(0) - 2\pi b\eta_1'(R). \tag{3.141}$$

Letting $n \to \infty$ and using the Beppo-Levi Theorem implies

$$\lim_{n\to\infty} \|(g(\upsilon_{c_n}) - g(\upsilon^*))\,\eta_1\|_{L^1(B_R)} = 0.$$

Thus υ^* is the solution of (3.136) with $c = c_+$.

Step 3 Nonexistence of a solution to problem (3.136) in the case $c > c_+(g)$. Suppose that such a solution υ_c exists. Because of uniqueness, it is a radial function, and $g(\upsilon_c) \in L^1(B_R)$. The function

$$r \mapsto w(r) - \frac{c}{2\pi}\ln(1/r),$$

satisfies $(rw'(r))' = rg(\upsilon_c)$ on $(0, R)$. Therefore $r \mapsto rw'(r)$ admits a limit when $r \to 0$. If the limit were not zero, say α, it would imply

$$w(r) = \alpha \ln(1/r)(1 + \circ(1)) \quad as\ r \to 0,$$

and

$$\Delta w = rg(\upsilon_c) - 2\pi c\delta_0,$$

contradiction. Thus $rw'(r) \to 0$ as $r \to 0$, and by integration,

$$\upsilon_c(r) = \frac{c}{2\pi}\ln(1/r)(1 + \circ(1)). \tag{3.142}$$

Then, for any $0 < \gamma < c$, there exists $R_\gamma \in (0, R]$ such that

$$\upsilon_c(r) \geq \frac{\gamma}{2\pi}\ln(1/r), \quad \text{in } (0, R_\gamma].$$

Thus $g(\upsilon_c) \geq g(\gamma/(2\pi)\ln(1/r))$. Put $a = 2\pi/\gamma$. Since $g(\upsilon_c) \in L^1(B)$, it implies

$$\int_0^\infty g(s)e^{-2as}ds < \infty \Longrightarrow 2a \geq a_+(g),$$

and finally $c \leq c_+(g)$, a contradiction.

Step 4 The relaxation phenomena when $c > c_+(g)$. For any n and any $\epsilon > 0$, the solution υ_n of (3.135) is bounded from below by the solution V_n of

$$\begin{aligned} -\Delta V_n + g_n(V_n) &= (c_+(g) - \epsilon)\delta_0 \quad && \text{in } \mathcal{D}'(B_R), \\ V_n &= b && \text{on } \partial B_R. \end{aligned} \tag{3.143}$$

Let $\tilde{\upsilon}$ be the limit of the υ_n. Then $\tilde{\upsilon}$ is a solution of

$$\begin{aligned} -\Delta\tilde{\upsilon} + g(\tilde{\upsilon}) &= 0 \quad && \text{in } B_R^*, \\ \tilde{\upsilon} &= b && \text{on } \partial B_R. \end{aligned} \tag{3.144}$$

Because V_n converges to $\upsilon_{c_+(g)-\epsilon}$, there holds $\tilde{\upsilon} \geq \upsilon_{c_+(g)-\epsilon}$. Letting $\epsilon \to 0$ finally yields to $\tilde{\upsilon} \geq \upsilon_{c_+}(g)$. Taking the same test function η_1 defined by (3.140), one obtains

$$\int_{B_R} (-\upsilon_n\Delta\eta_1 + g_n(\upsilon_n)\eta_1)\,dx = c\eta_1(0) - 2\pi b\eta_1'(R). \tag{3.145}$$

Using the fact that $\upsilon_n \leq \Psi$ (see Step 1) and Fatou's lemma,

$$\int_{B_R} g(\tilde{\upsilon})\eta_1 dx \leq \liminf_{n\to\infty}\int_{B_R} g_n(\upsilon_n)\eta_1 dx < \infty.$$

Thus $g(\tilde{v}) \in L^1(B_R)$. Since $\tilde{v} \in L^1(B_R)$, the distribution $T = -\Delta\tilde{v} + g(\tilde{v})$ has the point 0 for support, therefore there exist real numbers c_p, $(p \in \mathbb{N}^m)$ such that

$$T = \sum_{|p| \leq m} c_p D^p \delta_0.$$

Let $\zeta \in C_c^\infty(B)$ such that

$$(-1)^{|p|} D^p \zeta(0) = c_p, \quad \forall p \in \mathbb{N}^m, \ |p| \leq m,$$

and for $\epsilon > 0$, put $\zeta_\epsilon(x) = \zeta(x/\epsilon)$. Then

$$\int_B (-\tilde{v}\Delta\zeta_\epsilon + g(\tilde{v})\zeta_\epsilon)\, dx = \sum_{|p| \leq m} \frac{c_p^2}{\epsilon^{|p|}}. \tag{3.146}$$

But

$$\left| \int_B \tilde{v}\Delta\zeta_\epsilon dx \right| = \frac{1}{\epsilon^2} \left| \int_B \tilde{v}\Delta\zeta(x/\epsilon) dx \right| \leq \frac{C}{\epsilon^2} \int_0^{R\epsilon} \ln(1/s) s ds \leq C' \ln(1/\epsilon). \tag{3.147}$$

Comparing (3.146) and (3.147) implies $c_p = 0$ for any $|p| \geq 1$, from what is infered

$$-\Delta\tilde{v} + g(\tilde{v}) = c_0\delta_0 \quad \text{in } \mathcal{D}'(B). \tag{3.148}$$

By Step 3 and the inequality $\tilde{v} \geq \upsilon_{c_+}(g)$, one has $c_0 = c_+(g)$, which ends the proof. $\square$

Proof of Proposition 3.41. Assume such a u exists. By changing R, we can assume that $u \in L^1(\partial B_R)$ and that u is therefore the unique integrable function with $g(u) \in L^1(B_R)$ which satisfies

$$\begin{aligned} -\Delta u + g(u) = & \quad c\delta_0 + \nu \quad \text{in } \mathcal{D}'(B_R), \\ u & \quad \text{fixed on } \partial B_R. \end{aligned} \tag{3.149}$$

Put $g_n(r) = \min\{g(r), g(n)\}$, and let υ_n be the solution of

$$\begin{aligned} -\Delta\upsilon_n + g_n(\upsilon_n) = c\delta_0 & \qquad \text{in } \mathcal{D}'(B_R), \\ \upsilon_n = 0 & \qquad \text{on } \partial B_R, \end{aligned} \tag{3.150}$$

and v the one of

$$\begin{aligned} -\Delta v = \nu_+ & \qquad \text{in } \mathcal{D}'(B_R), \\ v = u_+ & \qquad \text{on } \partial B_R. \end{aligned} \tag{3.151}$$

Since $g(\upsilon_n + v) \geq g_n(\upsilon_n + v) \geq g_n(\upsilon_n)$, the function $U_n = \upsilon_n + v$ is a super-solution for Problem (3.149). Therefore $u \leq \upsilon_n + v$. Letting $n \to \infty$ and using Lemma 3.42 yields to

$$u \leq \upsilon_{c_+(g)} + v. \tag{3.152}$$

Writing again

$$u(r,\theta) = u(x) = \frac{c}{2\pi} \ln(1/|x|) + \omega(x),$$

then

$$-\Delta\omega = \nu - g(u) \Longrightarrow -\Delta\bar{\omega}(r) = \overline{(\nu - g(u))(r)},$$

where the overlining indicates the angular average. Because the measure ν has no atom and $g(u) \in L^1(B_R)$,

$$\int_0^r \overline{(\nu - g(u))(s)}ds \to 0, \quad \text{as } r \to 0.$$

Thus

$$\overline{u}(r) = \frac{c}{2\pi}\ln(1/r)(1 + \circ(1)).$$

In the same way

$$\overline{\omega}(r) = \circ(\ln(1/r)),$$

and, from Lemma 3.42-Step 2,

$$\overline{v}_{c_+(g)}(r) = v_{c_+(g)}(r) = \frac{c_+(g)}{2\pi}\ln(1/r)(1 + \circ(1)).$$

Since $c > c_+(g)$, this contradicts (3.152). $\square$

Proof of Theorem 3.40. By replacing λ by $\lambda - g(0)$, it is always possible to assume $g(0) = 0$. The measure λ admits the decomposition

$$\lambda = \sum_{j\in J} c_j\delta_{x_j} + \nu,$$

where $\{x_j\}_{j\in J}$ is the set of atoms of λ, and ν is the sum of a measure absolutely continuous with respect to the 2-dimensional Hausdorff measure and a singular measure without atom.

Step 1 We assume that λ is positive with compact support in Ω, and $c_j < c_+(g)$ for any $j \in J$. Let $\delta > 0$ as in Theorem 3.37, $J_1 = \{j \in J : c_j \geq \delta/2\}$ (with $\#(J_1) = K$), and $j_2 = J \setminus J'$. We denote

$$\lambda_\delta = \lambda - \sum_{j\in J_1} c_j\delta_{x_j}.$$

First, there exists a finite covering $\{\Omega_i\}_{i\in I}$ of Ω (with $\#(I) = N$) such that $\Omega_i \cap \Omega_{i'} = \emptyset$ if $i \neq i'$, and

$$\int_{\Omega_i} d\lambda_\delta < \delta. \tag{3.153}$$

This covering can be chosen such that any $\overline{\Omega}_i$ contains at most one x_j for $j \in J_1$, and actually $x_j \in \Omega_i$, we shall write $i = i(j)$ and this correspondence is one to one from J_1 into I. For such a x_j, there exists $\sigma_j > 0$ such that $\overline{B_{\sigma_j}(x_j)} \subset \Omega_{i(j)}$, and

$$\lim_{\sigma\to 0}\int_{B_\sigma(x_j)} d(\lambda - c_j\delta_{x_j}) = 0. \tag{3.154}$$

Let $R > 0$ be such that $\overline{\Omega} \subset B_R(x_j)$, $\forall j \in J_1$. For $0 < \sigma \leq \inf_{j\in J_1}\sigma_j$ and $i = i(j)$ for some $j \in J_1$, we set

$$\Omega_{i(j)} = \overline{B_\sigma(x_j)} \cup \Omega'_{i(j),\sigma}.$$

By Lemma 3.42-Step 1, each of the following equations admits a solution u_j,

$$\begin{aligned} -\Delta u_j + \frac{1}{2N} g(u_j) &= c_j \delta_{x_j} \quad \text{in } \mathcal{D}'(B_R(x_j)), \\ u_j &= 0 \quad \text{on } \partial B_R(x_j), \end{aligned} \tag{3.155}$$

for $j \in J_1$. Let $\Omega_{i,\sigma} = \{x \in \Omega_i : \operatorname{dist}(x, \Omega_i^c) > \sigma\}$. If $i \in I \setminus \{i(j) : j \in J_1\}$, we set $\lambda_{i,\sigma} = \chi_{\Omega_{i,\sigma}} \lambda_\delta$, and if $i = i(j)$ for some $j \in J_1$, we put $\lambda_{i,\sigma} = \chi_{\Omega'_{i,\sigma}} \lambda_\delta$. By Theorem 3.37 there exist functions $v_{i,\sigma}$ solutions of

$$\begin{aligned} -\Delta v_{i,\sigma} + \frac{1}{2N} g(v_{i,\sigma}) = \lambda_{i,\sigma} \quad & \text{in} \mathcal{D}'(\Omega), \\ v_{i,\sigma} = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{3.156}$$

for $i \in I$. Furthermore the u_j and $v_{i,\sigma}$ are respectively the limit of the $u_{j,n}$ and $v_{i,\sigma,n}$ solutions of

$$\begin{aligned} -\Delta u_{j,n} + \frac{1}{2N} g(u_{j,n}) = c_j \delta_{x_j} * \rho_n \quad & \text{in } \mathcal{D}'(B_R(x_j)), \\ u_{j,n} = 0 \qquad\qquad & \text{on } \partial B_R(x_j), \end{aligned} \tag{3.157}$$

and

$$\begin{aligned} -\Delta v_{i,\sigma,n} + \frac{1}{2N} g(v_{i,\sigma,n}) = \lambda_{i,\sigma} * \rho_n \quad & \text{in } \mathcal{D}'(\Omega), \\ v_{i,\sigma,n} = 0 \qquad\qquad & \text{on } \partial\Omega, \end{aligned} \tag{3.158}$$

where ρ_n is a positive radial and smooth convolution kernel with shrinking compact support. Hence, for n large enough and σ small enough, the support of the $c_j \delta_{x_j} * \rho_n$ and $\lambda_{i,\sigma} * \rho_n$ are all disjoint and included in $B_{\sigma/2}(x_j)$ or in $\Omega_{i,\sigma/2}$ (if $i \notin i(J_1)$), or in $\Omega'_{i(j),\sigma/2}$. Finally, $g(u_{j,n}) \to g(u_j)$ in $L^1(B_R(x_j))$ (easy to check from Lemma 3.42-Step 1) and $g(v_{i,\sigma,n}) \to g(v_{i,\sigma})$ in $L^1(\Omega)$, as $n \to \infty$ (by the proof of Theorem 3.34). Put

$$U_n = \sum_{j \in J_1} u_{j,n}, \quad U = \sum_{j \in J_1} u_j,$$

both quantities defined in $\overline{\Omega}$, and

$$V_n = \sum_{i \in I} v_{i,\sigma,n}, \quad V_\sigma = \sum_{i \in I} v_{i,\sigma}.$$

With the same convolution kernel ρ_n, we denote by $u_{n,\sigma}$ the solution to

$$\begin{aligned} -\Delta u_{\sigma,n} + g(u_{\sigma,n}) = \lambda_\sigma * \rho_n \quad & \text{in } \mathcal{D}'(\Omega), \\ u_{\sigma,n} = 0 \qquad\qquad & \text{on } \partial\Omega, \end{aligned} \tag{3.159}$$

where

$$\lambda_\sigma = \sum_{j \in J_1} c_j \delta_{x_j} + \sum_{i \in I \setminus i(J_1)} \chi_{\Omega_{i,\sigma}} \lambda_\delta + \sum_{i \in i(J_1)} \chi_{\Omega'_{i,\sigma}} \lambda_\delta.$$

As in the proof of Theorem 3.34, $u_{\sigma,n} \to u_\sigma$ in $L^1(\Omega)$ and a.e. in Ω, $g(u_{\sigma,n})$ is bounded in $L^1(\Omega)$, and $g(u_{\sigma,n}) \to g(u_\sigma)$ a.e. in Ω. Because

$$\begin{aligned}
-\Delta(U_n + V_{\sigma,n}) + g(U_n + V_{\sigma,n}) &= -\sum_{j\in J_1} \Delta u_{j,n} - \sum_{i\in I} \Delta v_{i,\sigma,n} + g(U_n + V_{\sigma,n}) \\
&\geq \sum_{j\in J_1} \left(-\Delta u_{j,n} + \frac{1}{2N} g(u_{j,n}) \right) \\
&\qquad + \sum_{i=1}^{N} \left(-\Delta v_{i,\sigma,n} + \frac{1}{2N} g(v_{i,\sigma,n}) \right) \\
&= \lambda_\sigma * \rho_n \qquad \text{in } \mathcal{D}'(\Omega),
\end{aligned} \tag{3.160}$$

and $U_n + V_{\sigma,n} \geq 0$ on $\partial\Omega$, one obtains

$$0 \leq u_{\sigma,n} \leq U_n + V_{\sigma,n}.$$

The estimate of the uniform integrability of $\{g(U_n + V_{\sigma,n})\}$ derives from the following argument : Let ω be a Borel subset of Ω and $\omega_i = \Omega_i \cap \omega$, $i \in I$. If $i \notin i(J_1)$ we can write

$$U_n + V_{\sigma,n} = v_{i,\sigma,n} + K(x), \quad \forall x \in \omega_i,$$

and, for σ fixed small enough, the function $x \mapsto K(x)$ is bounded uniformly with respect to n and $x \in \omega_i$, since the distance of the supports of the $\lambda_{i',\sigma} * \rho_n$ $(i' \neq i)$, and the $c_j \delta_{x_j} * \rho_n$ $(j \in J_1)$ to ω_i is larger or equal to $\sigma/2$. As in the proof of Theorem 3.34, we set

$$\theta_{n,i}(s) = \int_{\{x\in\omega_i : |(U_n + V_{n,\sigma})(x)| > s\}} dx,$$

and

$$\theta_{n,i}(s) \leq \int_{\{x\in\omega_i : v_{i,\sigma,n} + K(x)) > s\}} dx.$$

The proof of Theorem 3.34 applies : for $\epsilon > 0$ fixed, there exists $\delta > 0$, such that

$$|\omega_i| \leq \delta \Longrightarrow \int_{\omega_i} g(U_n + V_{n,\sigma}) dx < \epsilon/2N. \tag{3.161}$$

If $i = i(j)$ we put $\omega_i = \omega_i' \cup \omega_i''$, where $\omega_i' \subset \Omega'_{i(j),\sigma}$ and $\omega_i'' \subset B_\sigma(x_j)$. On ω_i' we write

$$U_n + V_{n,\sigma} = v_{i(j),\sigma,n} + K'(x),$$

and $K'(x)$ is bounded independently of n, thus (3.161) holds with ω_i' instead of ω_i. On ω_i'' there holds

$$U_n + V_{n,\sigma} = u_{i(j),n} + K''(x),$$

with $K''(x)$ bounded independently of n. Thus

$$g(U_n + V_{n,\sigma}) \leq g(u_{i(j),n} + K''(x)).$$

Because $g(u_{i(j),n}) \to g(u_{i(j)})$ in $L^1(B_R(x_{i(j)})))$ as $n \to \infty$, $g(u_{i(j),n} + k) \to g(u_{i(j)} + k)$ for any $k > 0$. Thus $\{g(u_{i(j),n} + k)\}$ is uniformly integrable. The same holds with $\{g(u_{i(j),n} + K"(x))\chi_{B_\sigma(x_{i(j)})}\}$, if we take $k \geq K''$. Finally (3.161) holds with ω_i'' instead of ω_i. Consequently,

$$\forall \omega \subset \Omega,\ \omega \text{ Borel },|\omega| \leq \delta \Longrightarrow \int_\omega g(u_{n,\sigma})dx \leq \int_\omega g(U_n + V_{n,\sigma})dx < \epsilon. \tag{3.162}$$

We conclude by Vitali's theorem that $g(u_{n,\sigma}) \to g(u_\sigma)$ in $L^1(\Omega)$, thus u_σ is the solution of

$$\begin{aligned} -\Delta u_\sigma + g(u_\sigma) = \lambda_\sigma \quad &\text{in } \mathcal{D}'(\Omega), \\ u_\sigma = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{3.163}$$

In particular there holds

$$\int_\Omega (u_\sigma + g(u_\sigma)\eta_1 dx = \int_\Omega \eta_1 d\lambda_\sigma,$$

if we take

$$\begin{aligned} -\Delta\eta_1 &= 1 \quad \text{in } \Omega, \\ \eta_1 &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Letting $\sigma \to 0$, u_σ increases to u and

$$\int_\Omega (u + g(u)\eta_1 dx = \int_\Omega \eta_1 d\lambda. \tag{3.164}$$

From this integrability property it follows that u is the solution of (3.132).

Step 2 The case of a general positive bounded measure. We perform a double truncation, replacing λ by λ_n ($n \in \mathbb{N}_*$), by putting

$$\lambda_n = \sum_{j \in J_{c_+}} (c_+(g) - n^{-1})\delta_{x_j} + \chi_{\Omega_n} \left(\sum_{j \in J \setminus J_{c_+}} c_{x_j} \delta_{x_j} \nu \right),$$

where $J_{c_+}\{j \in J : c_j = c_+(g)\}$, ν is the non-atomic part of λ, and $\Omega_n = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/n\}$. If u_n is the solution corresponding to (3.132), with λ replaced by λ_n, the sequence $\{u_n\}$ is increasing and converges to some integrable function u. As in Step 1, we conclude, by Beppo-Levi's theorem and using Equality (3.164) with λ_n and u_n instead of λ and u, that $g(u_n)$ converges to $g(u)$ a.e. and in $L^1(\Omega; \rho_{\partial\Omega})$ and (3.164) still holds at the limit. Furthermore $g(u) \in L^1(\Omega)$ by Proposition 3.2.

Step 3 The case of a general bounded measure. If $\lambda = \lambda_+ - \lambda$ is a bounded measure, subcritical with respect to g, we have

$$\lambda_+ = \sum_{j \in J_+} c_j \delta_{x_j} + \nu_+,$$

$$-\lambda_- = \sum_{j\in J_-} c'_j \delta_{x'_j} - \nu_-,$$

where $\{(c_j, x_j)_{j\in J_+}\}$ (resp. $\{(c_j, x'_j)_{j\in J_-}\}$) is the set of positive atoms $c_j > 0$ (resp. $c'_j < 0$). We truncate the measures λ_+ and λ_- as in Step 2, introduce the coverings $\{\Omega_i\}$ and $\{\tilde{\Omega}_i\}$ and the separation parameter σ and construct the sets of solutions u_j^+, $\upsilon_{j,\sigma}^+$, u_j^- and $\upsilon_{j,\sigma}^-$ such that

$$\begin{array}{rcll} -\Delta u_j^+ + \dfrac{1}{2N} g(u_j^+) &=& c_j\delta_{x_j} & \text{in } \mathcal{D}'(B_R(x_j)), \\ u_j^+ &=& 0 & \text{on } \partial B_R(x_j), \end{array}$$

$$\begin{array}{rcll} -\Delta \upsilon_{j,\sigma}^+ + \dfrac{1}{2N} g(\upsilon_{j,\sigma}^+) &=& \lambda_{+\,i,\sigma} & \text{in } \mathcal{D}'(\Omega), \\ \upsilon_{j,\sigma}^+ &=& 0 & \text{on } \partial\Omega, \end{array}$$

$$\begin{array}{rcll} -\Delta u_j^- + \dfrac{1}{2N} g(u_j^-) &=& c'_j\delta_{x'_j} & \text{in } \mathcal{D}'(B_R(x'_j)), \\ u_j^- &=& 0 & \text{on } \partial B_R(x'_j), \end{array}$$

and

$$\begin{array}{rcll} -\Delta \upsilon_{j,\sigma}^- + \dfrac{1}{2N} g(\upsilon_{j,\sigma}^-) &=& \lambda_{-\,i,\sigma} & \text{in } \mathcal{D}'(\Omega), \\ \upsilon_{j,\sigma}^- &=& 0 & \text{on } \partial\Omega, \end{array}$$

and their approximations $u_{j,n}^+$, $\upsilon_{j,\sigma,n}^+$, $u_{j,n}^-$ and $\upsilon_{j,\sigma,n}^-$. We also construct u_n solution of (3.159). As in Step 1, we obtain

$$U_{-\,n} + V_{-\,\sigma,n} \leq u_{\sigma,n} \leq U_{+\,n} + V_{+\,\sigma,n},$$

where $U_{+\,n}$, $V_{+\,\sigma,n}$, $U_{-\,n}$, $V_{-\,\sigma,n}$ are defined as U_n and $V_{\sigma,n}$ as in Step 1, from the $u_{j,n}^+$, $\upsilon_{j,\sigma,n}^+$, $u_{j,n}^-$ and $\upsilon_{j,\sigma,n}^-$. Because

$$g(U_{-\,n} + V_{-\,\sigma,n}) \leq g(u_n) \leq g(U_{+\,n} + V_{+\,\sigma,n}),$$

and the sets of functions $\{g(U_{-\,n}+V_{-\,\sigma,n})\}$ and $\{g(U_{+\,n}+V_{+\,\sigma,n})\}$ are uniformly integrable from Step 1, the same property is shared by the set $\{g(u_n)\}$. We conclude by the Vitali Theorem as in Step 1, letting $n \to \infty$ and $\sigma \to 0$. The other convergences, as in Step 2, follow by the same uniform integrability arguments and the monotonity. $\square$

The general approximation-relaxation result of [94] is the following.

Theorem 3.43 *Let g be a continuous nondecreasing function with finite exponential orders of growth at plus and minus infinity, and $\lambda \in \mathfrak{M}^b(\Omega)$ with decomposition*

$$\lambda = \lambda^* + \lambda_s + \sum_{j\in J} c_j\delta_{x_j},$$

λ^*, λ_s *being respectively the absolute continuous part and the singular non-atomic part of* λ. *Let*

$$J^+ = \{j \in J : c_j > c_+(g)\}, \text{ and } J^- = \{j \in J : c_j < c_-(g)\},$$

ρ_n *be a regularizing kernel and* u_n *the solution of*

$$\begin{aligned} -\Delta u_n + g(u_n) &= \lambda * \rho_n \quad \text{in } \mathcal{D}'(\Omega), \\ u_n &= 0 \qquad\quad \text{on } \partial\Omega. \end{aligned} \tag{3.165}$$

Then $u_n \to u$ *in* $L^1(\Omega)$ *where* u *is the solution of*

$$\begin{aligned} -\Delta u + g(u) &= \lambda^r \quad \text{in } \mathcal{D}'(\Omega), \\ u &= 0 \quad\ \text{on } \partial\Omega, \end{aligned} \tag{3.166}$$

and

$$\lambda^r = \lambda^* + \lambda_s + \sum_{j \in J \setminus \{J^+ \cup J^-\}} c_j \delta_{x_j} + \sum_{j \in J^+} c_+(g) \delta_{x_j} + \sum_{j \in J^-} c_-(g) \delta_{x_j}.$$

The proof of this results follows by a combination of the arguments in Proposition 3.41 and Theorem 3.40.

4 Semilinear equations with source term

4.1 The basic approach

The equation under consideration is written under the form

$$\begin{aligned} Lu &= g(x,u) + \lambda \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

where Ω is a domain in $\mathbb{R}^n$, L an elliptic operator defined in Ω, g a continuous function defined in $\mathbb{R} \times \Omega$ and λ a Radon measure in Ω. The following general result plays an important role in proving existence of solutions in presence of supersolutions and subsolutions (see e.g. [82], [87]).

Theorem 4.1 *Let* $\Omega \subset \mathbb{R}^n$ *be any domain,* L *a second order elliptic operator defined by the expression (2.1) with locally Lipschitz continuous coefficients. We assume that for any compact subset* $K \subset \Omega$ *there exists* $\alpha_K > 0$ *such that*

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \alpha_K \sum_{i=1}^{n} \xi_i^2, \quad \forall x \in K,\ \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{4.2}$$

Let h^*, $h^\dagger \in C(\Omega \times \mathbb{R})$ *be such that* $r \mapsto h^*(x,r)$ *is nondecreasing for every* $x \in \Omega$, *and* $(x,r) \mapsto h^\dagger(x,r)$ *is locally Lipschitz continuous with respect to the* r *variable, uniformly* *when the x variable stays in a compact subset of Ω, and put $h = h^* + h^\dagger$. If there exist two $C(\Omega) \cap W^{1,2}_{loc}(\Omega)$-functions u_* and u^* satisfying*

$$\begin{array}{llr} (i) & Lu_* + h(x,u_*) \geq 0 \quad \text{in } \Omega, & \\ (ii) & Lu^* + h(x,u^*) \leq 0 \quad \text{in } \Omega, & (4.3) \\ (iii) & u_* \leq u^* \qquad\qquad\quad \text{in } \Omega, & \end{array}$$

where the equations are understood in the weak sense, then there is a $C^1(\Omega)$-function u which satisfies

$$\begin{array}{ll} (i) & Lu + h(x,u) = 0 \quad \text{in } \Omega, \\ (ii) & u_* \leq u \leq u^* \qquad\quad \text{in } \Omega. \end{array} \tag{4.4}$$

The following construction is at the origin of most of the methods for solving semilinear equations with reaction source term : if Ω is a bounded domain in $\mathbb{R}^n$ with a C^2 boundary and L the elliptic operator defined by (2.1) satisfying condition (H), if u is an integrable function solution of (4.1) with $\lambda \in \mathfrak{M}(\Omega;\rho_{\partial\Omega})$ such that $g(.,u) \in L^1(\Omega;\rho_{\partial\Omega}dx)$, there holds

$$u(x) = \int_\Omega G^\Omega_L(x,y)g(y,u(y)dy + \int_\Omega G^\Omega_L(x,y)d\lambda(y), \quad a.e. \text{ in } \Omega. \tag{4.5}$$

Theorem 4.2 *Assume $g(x,0) = 0$, $r \mapsto g(x,r)$ is nondecreasing for any $x \in \Omega$ and $\lambda \in \mathfrak{M}(\Omega;\rho_{\partial\Omega})$ satisfies $\mathbb{G}^\Omega_L(\lambda) \geq 0$. If there exists some $v \in L^1(\Omega)$, $v \geq 0$ such that $g(.,v) \in L^1(\Omega;\rho_{\partial\Omega}dx)$ and*

$$v \geq \mathbb{G}^\Omega_L(g(.,v) + \mathbb{G}^\Omega_L(\lambda), \tag{4.6}$$

there exists a positive solution u to Problem (4.1).

Proof. The sequence $\{u_n\}_{n\in\mathbb{N}}$ defined by $u_0 = 0$ and

$$u_{n+1} = \mathbb{G}^\Omega_L(g(.,u_n) + \mathbb{G}^\Omega_L(\lambda), \quad \forall n \in \mathbb{N}, \tag{4.7}$$

is nondecreasing, as soon as $\mathbb{G}^\Omega_L(g(.,u_n)$ exists, but the u_n are well defined because it is easy to prove by induction that there holds

$$0 = u_0 \leq u_1 \leq u_2 \leq \ldots \leq u_n \leq v. \tag{4.8}$$

Therefore there exists $u = \lim_{n\to\infty} u_n$ which satisfies $0 \leq u \leq v$, $u \in L^1(\Omega)$, $g(.,u) \in L^1(\Omega;\rho_{\partial\Omega}dx)$ and

$$u = \mathbb{G}^\Omega_L(g(.,u) + \mathbb{G}^\Omega_L(\lambda). \tag{4.9}$$

This means that u is a solution of (4.1). □

4.2 The convexity method

The convexity method due to Baras and Pierre [10] applies to a large variety of problems which contains Problem (4.1).

4.2.1 The general construction

Let (U, μ) be a positive measured space with a σ-finite measure μ. We assume that $\{K_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable subsets of U such that

$$\mu(K_n) < \infty, \quad \forall n \in \mathbb{N}, \quad \bigcup_{n\geq 0} K_n = U. \tag{4.10}$$

We denote by $L_+(U)$ (resp. $L_+(U\times U)$) the space of μ-measurable (resp. $\mu\otimes\mu$-measurable) functions with value in $[0,\infty]$. We consider a kernel $N \in L_+(U \times U)$ and a fuction $j : U \times \mathbb{R} \mapsto [0,\infty]$, $\mu \otimes dx$-measurable such that

$$\begin{aligned} &(i)\; r \mapsto j(x,r) \text{ is nondecreasing, convex and l.s.c., for almost all } x \in U,\\ &(ii)\; j(x,0) = 0, \text{ a.e. in } U. \end{aligned} \tag{4.11}$$

The conjugate function j^*, defined by

$$j^*(x,r) = \sup_{\alpha\in\mathbb{R}}(r\alpha - j(x,r)) \tag{4.12}$$

satisfies (4.11). If $u \in L_+(U)$,

$$j(u)(x) = \begin{cases} j(x,u(x)) & \text{if } u(x) < \infty,\\ \lim\limits_{r\to\infty} j(x,r) & \text{if } u(x) = \infty. \end{cases} \tag{4.13}$$

If $h \in L_+(U)$ we set

$$\mathbb{N}(h)(x) = \int_U N(x,y)h(y)d\mu(y),$$

and

$$\mathbb{N}^*(h)(y) = \int_U N(x,y)h(x)d\mu(x).$$

Notice that these two quantities are positive or infinite. All the $L^p(U)$-spaces $(1 \leq p \leq \infty)$ are relative to the measure μ. We denote by $L^p_+(U)$ their positive cones,

$$L^\infty_c(U) = \{h \in L^\infty(U) : \exists n \in \mathbb{N} \text{ s.t. } h(x) = 0, \text{ a.e. in } U \setminus K_n\}, \tag{4.14}$$

and $L^\infty_{c+}(U) = L^\infty_c(U) \cap L_+(U)$. Being given $f \in L_+(U)$, the general problem lies in finding $u \in L_+(U)$ such that

$$u = \mathbb{N}(j(u)) + f. \tag{4.15}$$

Multiplying (4.15) by h and integrating over U implies

$$\begin{aligned} \int_U fhd\mu = \int_U (u - \mathbb{N}^*(j(u)))hd\mu &= \int_U (uh - j(u)\mathbb{N}^*(h))d\mu \\ &= \int_U \mathbb{N}^*(h)\left(u\frac{h}{\mathbb{N}^*(h)} - j(u)\right)d\mu \\ &\leq \int_U j^*\left(\frac{h}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h)d\mu, \end{aligned} \tag{4.16}$$

provided $uh \in L^1(U)$. Therefore a necessary condition for existence of a solution to Equation (4.15) is

$$\int_U fhd\mu \leq \int_U j^*\left(\frac{h}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h)d\mu, \quad \forall h \in L^\infty_{c+}(U) \text{ such that } uh \in L^1(U). \tag{4.17}$$

Under a very mild additional assumption, this condition is also sufficient. Being given $C \geq 1$ and $h \in L^\infty_{c+}(U)$, we denote

$$F_C(h) = \begin{cases} \displaystyle\int_U j^*\left(\frac{h}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h)d\mu \text{ if } \frac{h}{\mathbb{N}^*(h)} < \infty \text{ a.e.} \\ \qquad\qquad\qquad\qquad \text{and } j^*\left(\dfrac{h}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h) \in L^1(U), \\ +\infty \qquad\qquad \text{if not.} \end{cases} \tag{4.18}$$

with the convention $h(x)/\mathbb{N}^*(h)(x) = 0$ if $h(x) = \mathbb{N}^*(h)(x) = 0$. If $C = 1$, $F_1 = F$. We put

$$X = \{h \in L^\infty_c(U) : F(h) < \infty\},$$

and

$$\hat{X} = \{h \in L^\infty_c(U) : \exists C > 1 \text{ s.t. } F_C(h) < \infty\}.$$

In the sequel we adopt the convention $uh(x) = 0$ if $h(x) = 0$ and $u(x) = \infty$. The main existence result is as follows.

Theorem 4.3 *Let $f \in L_+(U)$. The following problem*

$$\begin{aligned} &(i) \quad u \in L_+(U), \quad u(x) = \mathbb{N}(j(u))(x) + f(x) \quad \mu\text{-a.e. in } U, \\ &(ii) \quad uh \in L^1(U), \quad \forall h \in \hat{X}, \end{aligned} \tag{4.19}$$

admits a solution if and only if

$$\int_U fhd\mu \leq F(h), \quad \forall h \in \hat{X}. \tag{4.20}$$

Scheme of the proof. For $\gamma \in (0,1)$ we introduce the sequence $\{u_n\}$ defined by $u_0 = \gamma f$ and

$$u_{n+1} = \gamma\left(\mathbb{N}(j(u_n)) + f\right), \quad \forall n \in \mathbb{N}. \tag{4.21}$$

Step 1 We claim that

$$\int_U u_{n+1}hd\mu \leq \frac{\gamma}{1-\gamma}F(h), \quad \forall h \in \hat{X}. \tag{4.22}$$

For $1 < C < 1/\gamma$, and $h \in \hat{X}$ such that $F_C(h) < \infty$, we suppose that there exists some $\psi \in L_+(U)$ such that

$$\psi(x) = \max\left\{\frac{1}{C}j'(u_n)(x)\mathbb{N}^*(\psi)(x), h(x)\right\}. \tag{4.23}$$

It follows from (4.21),

$$\int_U u_{n+1}\psi d\mu = \gamma \int_U j(u_n)\mathbb{N}^*(\psi)d\mu + \gamma \int_U f\psi d\mu. \tag{4.24}$$

By assumption (4.20)

$$\begin{aligned}
\int_U f\psi d\mu \le F_C(\psi) &\le \int_U j^*\left(\frac{\max\{j'(u_n)\mathbb{N}^*(\psi), Ch\}}{\mathbb{N}^*(\psi)}\right)\mathbb{N}^*(\psi)d\mu \\
&\le \int_U \max\left\{j^*\left(j'(u_n)\mathbb{N}^*(\psi)\right), j^*\left(\frac{Ch}{\mathbb{N}^*(\psi)}\right)\mathbb{N}^*(\psi)\right\} d\mu.
\end{aligned}$$

Since $\psi \ge h$, one has $\mathbb{N}^*(\psi) \ge \mathbb{N}^*(h)$. By convexity $j^*(\alpha r) \le \alpha j^*(r)$, $\forall r \ge 0$, $\forall \alpha \in [0,1]$, therefore

$$j^*\left(\frac{Ch}{\mathbb{N}^*(\psi)}\right)\mathbb{N}^*(\psi) \le j^*\left(\frac{Ch}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h).$$

By definition $j^*(j'(u_n)) = u_n j'(u_n) - j(u_n)$. Thus, returning to (4.24) implies

$$\int_U u_{n+1}\psi d\mu \le \gamma \int_U j(u_n)\mathbb{N}^*(\psi)d\mu + \gamma \int_U \left(u_n j'(u_n) - j(u_n)\right)\mathbb{N}^*(\psi)d\mu + \gamma F_C(h).$$

By combining this inequality with the definition of ψ, one derives

$$\int_U u_{n+1}\psi d\mu \le \gamma \int_U u_n \psi d\mu + \gamma F_C(h).$$

Because $u_{n+1} \ge u_n$ and $\psi \ge h$, we obtain

$$\int_U u_{n+1} h d\mu \le \int_U u_{n+1}\psi d\mu \le \frac{\gamma}{1-\gamma C} F_C(h).$$

Letting $C \to 1$, (4.22) follows.

Step 2 Convergence. Letting $n \to \infty$, u_n increases and converges to some u_γ which satisfies

$$\begin{aligned}
&(i) \quad u_\gamma \in L_+(U), \quad u_\gamma = \gamma\left(\mathbb{N}(j(u_\gamma)) + f\right) \quad \text{in } U, \\
&(ii) \quad u_\gamma h \in L^1(U), \quad \forall h \in \hat{X},
\end{aligned} \tag{4.25}$$

This implies in particular

$$\int_U u_\gamma h d\mu = \gamma \int_U j(u_\gamma)\mathbb{N}^*(h)d\mu + \gamma \int_U f h d\mu, \quad \forall h \in \hat{X}.$$

Let $C > 1$ such that $F_C(h) < \infty$, then

$$\gamma \int_U \left(u_\gamma V \frac{h}{\mathbb{N}^*(h)} - j(u_\gamma)\right)\mathbb{N}^*(h)d\mu = (\gamma C - 1)\int_U u_\gamma h d\mu + \gamma \int_U f h d\mu,$$

and consequently

$$\int_U u_\gamma h d\mu \le \frac{\gamma}{\gamma C - 1} F_C(h). \tag{4.26}$$

Since the correspondence $\gamma \mapsto u_\gamma$ is increasing and, for almost all $x \in U$, $r \mapsto j(x,r)$ is continuous on the left, we can let $\gamma \to 1$ in (4.26) and (4.25)-(i) and deduce that the function $u = \lim_{\gamma \to 1} u_\gamma$ is a solution to problem (4.19).

Step 3 Justification. The difficulties in the above proof are of two kinds :

(1) It is not clear that $u_n < \infty$ on a set of positive measure. It is even not known if $u_0 = \gamma f$ satisfies $j(u_0) < \infty$ a.e. in U. To go arround this difficulty we approximate $j(u_n)$, formally equal to $u_n j'(u_n) - j^*(j'(u_n))$, by $u_n\beta_n - j^*(\beta_n)$ where the $\{\beta_n\}$ is an increasing sequence of regular enough fonctions converging to $j'(u_n)$.

(2) The existence of $\psi \in \hat{X}$ has to be proven.

The full construction, which is extremely technical, is performed in [10]. $\square$

In the presence of a subsolution v to Problem (4.19) it is possible to relax the assumption on the sign of f and to produce a signed solution u. More precisely, we assume that there exists a measurable function v such that

$$\begin{aligned} &(i) \quad v \in L^1(K_n) \text{ and } \quad N(.,.)j(v)(.) \in L^1(K_n \times U), \quad \forall n \in \mathbb{N}, \\ &(ii) \quad v(x) \leq \mathbb{N}(j(v))(x) + f(x) \quad \mu\text{-a.e. in } U, \end{aligned} \tag{4.27}$$

If $j : U \times \mathbb{R} \mapsto (-\infty, \infty]$ is a measurable function which satisfies (4.11), we introduce j_v^* and $\hat{X}_v$:

$$j_v^*(x,r) = \sup_{\alpha \geq v(x)} (r\alpha - j(x,\alpha),$$

and

$$\hat{X}_v = \left\{ h \in L_c^\infty(U) : \exists C > 1 \text{ s.t. } j_v^*\left(\frac{Ch}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h) \in L^1(U) \right\}.$$

Corollary 4.4 *There exists a measurable function $u : U \mapsto (-\infty, \infty]$ satisfying*

$$\begin{aligned} &(i) \quad u \geq v, \quad u(x) = \mathbb{N}(j(u))(x) + f(x) \quad \mu\text{-a.e. in } U, \\ &(ii) \quad uh \in L^1(U), \quad \forall h \in \hat{X}_v, \end{aligned} \tag{4.28}$$

if and only if

$$\int_U fhd\mu \leq \int_U j_v^*\left(\frac{Ch}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h)d\mu, \quad \forall h \in \hat{X}_v. \tag{4.29}$$

Proof. Put $w = u - v$ and define $\tilde{j}$ by

$\tilde{j}(x,r) = 0, \quad \forall (x,r) \in \Omega \times \mathbb{R}_-,$

$\tilde{j}(x,r) = j(x, r + v(x)) - j(x, v(x)) \quad \text{if } j(x,v(x)) < \infty \text{ and } r > 0,$

$\tilde{j}(x,r) = \infty \quad \text{if } j(x,v(x)) = \infty \text{ and } r > 0.$

Thus $\tilde{j}$ takes nonnegative values and satisfies (4.11). Moreover (4.28) is equivalent to

$$\begin{aligned} &(i) \quad w \in L_+(U), \quad w = \mathbb{N}(\tilde{j}(w)) + f + \mathbb{N}(j(v)) - v \quad \mu\text{-a.e. in } U, \\ &(ii) \quad wh \in L^1(U), \quad \forall h \in \hat{X}_v. \end{aligned} \tag{4.30}$$

Since

$$\tilde{j}^*(x,r) = j_v(x,r) + j(v(x)) - rv(x) \quad \text{if } j(x,v(x)) < \infty,$$
$$\tilde{j}^*(x,r) = 0 \quad \text{if } j(v) = \infty,$$

for any $h \in L_c^\infty(U)$, there holds

$$\tilde{j}^*\left(\frac{Ch}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h) = j_v^*\left(\frac{Ch}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h) + j(v)\mathbb{N}^*(h) - Chv, \tag{4.31}$$

μ-a.e. on $\{x \in U : j(v)(x) < \infty\}$. Therefore

$$\tilde{j}^*\left(\frac{Ch}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h) \in L^1(U) \Longleftrightarrow j^*\left(\frac{Ch}{\mathbb{N}^*(h)}\right)\mathbb{N}^*(h) \in L^1(U). \tag{4.32}$$

The proof of Corollary 4.4 follows from Theorem 4.3 applied to Problem (4.30). □

4.2.2 Application to elliptic semilinear equations

Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2 boundary, L an elliptic operator defined by (2.1) satisfying (H) and $j : \Omega \times \mathbb{R} \mapsto [0,\infty]$ a measurable function (for the $(n+1)$-dimensional Hausdorff measure) such that $j(x,r) = 0$, for almost all $x \in \Omega$ and every $r \leq 0$. The function $r \mapsto j(x,r)$ is also assumed to be convex, nondecreasing and l.s.c., thus it fulfills assumption (4.11). If $\lambda \in \mathfrak{M}_+(\Omega; \rho_{\partial\Omega})$, $f = \mathbb{G}_L^\Omega(\lambda) \in L^1(\Omega)$. We denote by

$$Y(L) = \{\xi \in C_c^{1,L}(\overline{\Omega})\} : L^*\xi \in L_c^\infty(\Omega) \cap L_+(\Omega), \tag{4.33}$$

the space C^1-functions ξ vanishing on $\partial\Omega$ such that $L^*\xi$ has compact support and is essentially bounded. Notice that the elements of $Y(L)$ are nonnegative by the maximum principle.

Theorem 4.5 *Assume there exist some $C > 1$ and $\xi_0 \in Y(L)$, $\xi \neq 0$, such that*

$$j^*\left(C\frac{L^*\xi_0}{\xi_0}\right) \in L^1(\Omega). \tag{4.34}$$

If $\lambda \in \mathfrak{M}_+(\Omega; \rho_{\partial\Omega})$, there exists at least one $u \in L^1_{loc}(\Omega)$ such that $\mathbb{G}_L^\Omega(j(u)) \in L^1_{loc}(\Omega)$ and

$$u = \mathbb{G}_L^\Omega(j(u)) + \mathbb{G}_L^\Omega(\lambda) \in L^1(\Omega), \text{ a.e. in } \Omega, \tag{4.35}$$

if and only if

$$\int_\Omega \xi d\lambda \leq \int_\Omega j^*\left(\frac{L^*\xi}{\xi}\right), \quad \forall \xi \in Y(L). \tag{4.36}$$

Moreover, if $\mu \geq 0$, there exists at least one positive solution.

Proof. We put $\mu = dx$, the n-dimensional Hausdorff measure, and

$$N(x,y) = G_L^\Omega(x,y), \quad \forall (x,y) \in \Omega \times \Omega,\, x \neq y.$$

Let v be defined by

$$v(x)=\begin{cases}0 & \text{if } f(x)\geq 0,\\ f(x) & \text{if } f(x)\leq 0.\end{cases}$$

Thus $v\in L^1(\Omega)$, $N^*(j(v))\equiv 0$ and (4.27) holds. Furthermore $j_v^*=j^*$ on $[0,\infty)$, $\hat{X}_v=\hat{X}\neq\{0\}$, because of (4.34). If it exists, any solution u of (4.35) satisfies $u\geq v$, thus this problem is equivalent to

$$\begin{cases}u\geq v, \quad u=\mathbb{N}(j(u))+f,\\ u\in L^1_{loc}(\Omega).\end{cases}$$

If $\xi\in Y(L)$, we put $h=L^*\xi$, which means equivalently

$$\xi=\mathbb{G}^{\Omega}_{L^*}(h)=\mathbb{N}^*(h).$$

By Corollary 4.4 there exists a measurable function u which satisfies $u=\mathbb{N}(j(u))+f$, $u\geq v$ and $uh\in L^1(\Omega)$, for every $h\in\hat{X}$. By (4.34), $uL^*\xi_0\in L^1(\Omega)$, then $u(x_0)$ is finite at least for one $x_0\in\Omega$, thus $N(x_0,.)j(u)(.)\in L^1(\Omega)$, by the equation. For any compact $K\subset\Omega$ and any compact neighborhood K_0 of $K\cup\{x_0\}$, there exists a constant C such that

$$G^{\Omega}_{L^*}(x,y)\leq CG^{\Omega}_{L^*}(x_0,y),\quad \forall(x,y)\in K\times(\Omega\setminus K_0).$$

Therefore

$$\int_K\int_{\Omega\setminus K_0}N(x,y)j(y,u(y))dydx\leq C\,|K|\int_{\Omega}N(x_0,y)j(y,u(y))dy<\infty,$$

from which it is infered that $\mathbb{N}(j(u))\in L^1_{loc}(\Omega)$, since K is arbitrary. Furthermore $u\in L^1_{loc}(\Omega)$, from the equation. $\square$

When $j(x,r)=r_+^q$, for some $q>1$, the result is as follows.

Corollary 4.6 *Let $q>1$, $\lambda\in\mathfrak{M}(\Omega;\rho_{\partial\Omega})$ and $\sigma>0$. Then there exists a function $u\in L^1_{loc}(\Omega)$ such that $\mathbb{G}^{\Omega}_L(u_+^q)\in L^1_{loc}(\Omega)$ satisfying*

$$\begin{aligned}Lu&=u_+^q+\sigma\lambda \quad \text{in } \Omega,\\ u&=0 \qquad\qquad \text{on } \partial\Omega,\end{aligned}\tag{4.37}$$

if and only if

$$\sigma\int_{\Omega}\xi d\lambda\leq\frac{\gamma-1}{\gamma\gamma'}\int_{\Omega}\frac{(L^*\xi)^{q'}}{\xi^{q'-1}},\quad \forall\xi\in Y(L),\tag{4.38}$$

where $q'=q/(q-1)$. Furthermore u is nonnegative if $\mathbb{G}^{\Omega}_L(\lambda)$ is so.

Condition (4.38) has two meanings : the first one is that the positive part of λ should not be too large, whatever is $q>1$, the second is that if q is above some critical value, measure λ should not be too concentrated. This concentration is expressed in terms of Bessel capacities as for equations with absorption. If we assume for example that $\lambda=\lambda_+-\lambda_-$ is a L^p-function, there holds,

Corollary 4.7 *Let $q > 1$, $\lambda = \lambda_+ - \lambda_- \in L^p(\Omega)$ Then there exists a function $u \in L^1_{loc}(\Omega)$ solution of Problem (4.37) for $\sigma > 0$, small enough, if*
(i) $n = 1, 2$ and $1 < q$, or $n \geq 3$ and $1 < q < n/(n-2)$,
or
(ii) $n \geq 3$, $q > n/(n-2)$ and $\lambda_+ \in L^p(\Omega)$ with $p \geq n(q-1)/2q$,
or
(ii) $n \geq 3$, $q = n/(n-2)$ and $\lambda_+ \in L^p(\Omega)$ with $p > 1$.

Proof. Only condition (4.36) is to be checked. If $\xi \in Y(L)$, we define w by

$$L^*\xi = w^{1/q'}\xi^{1/q}. \tag{4.39}$$

If $\dfrac{1}{p} + \dfrac{1}{\gamma} \leq 1$, there holds

$$\int_\Omega \xi d\lambda \leq \int_\Omega \xi d\lambda_+ \leq C\|\lambda_+\|_{L^p}\|\xi\|_{L^\gamma}. \tag{4.40}$$

If we assume

$$\frac{1}{s} \leq \frac{1}{\gamma} + \frac{2}{n}, \quad or \ \frac{1}{s} < \frac{2}{n}, \quad if\ \gamma = \infty, \tag{4.41}$$

it follows, by (4.39) and the Gagliardo and Sobolev inequalities,

$$\|\xi\|_{L^\gamma} \leq C\|\xi\|_{W^{2,s}} \leq C\|\Delta\xi\|_{L^s} \leq C\left(\int_\Omega w^{s/q'}\xi^{s/q}dx\right)^{1/s}.$$

for any $1 < s < \infty$. Furthermore, if

$$s \leq q', \tag{4.42}$$

one gets

$$\|\xi\|_{L^\gamma} \leq \left(\int_\Omega wdx\right)^{1/q'}\left(\int_\Omega \xi^{sq'/q(q'-s)}\right)^{(q'-s)/q's}.$$

If

$$\gamma \geq sq'/q(q'-s), \tag{4.43}$$

we derive

$$\|\xi\|_{L^\gamma} \leq C\int_\Omega wdx.$$

By combining this inequality with (4.40), it is infered

$$\int_\Omega \xi d\lambda \leq C\int_\Omega wdx.$$

In order to get (4.41), (4.42), (4.43), we choose $\gamma = \infty$, $s < n/2$ if $n = 1, 2$ or $n \geq 3$. We take $\gamma < \infty$ and s such that equality holds in (4.41), if $n \geq 3$, $q > n/(n-2)$, and $p \geq n(q-1)/2q$. □

The next result expresses the condition of concentration which allows a measure to be admissible in Problem (4.37).

Proposition 4.8 *Let $\lambda_\sigma = \sigma\lambda$ be a positive measure with compact support satisfying (4.38). Then there exists $k = k(q, n, \lambda_\sigma)$ such that*

$$\lambda_\sigma(K) \leq kC_{2,q'}(K), \quad \forall K \, compact \ , \ K \subset \Omega. \tag{4.44}$$

Proof. We first notice that (4.38) implies

$$\int_\Omega v d\lambda_\sigma \leq \frac{q-1}{q^{q'}} \int_\Omega \frac{|L^* v|^{q'}}{v^{q'-1}} dx, \quad \forall v \in C_c^\infty(\Omega), \ v \geq 0. \tag{4.45}$$

Indeed, if $v \geq 0$ belongs to $C_c^\infty(\Omega)$, we apply (4.38) to $\xi = \mathbb{G}_{L^*}^\Omega(|L^* v|)$ which is larger than v by the maximum principle. We replace v by $v^{2q'}$ in (4.45). Since

$$\begin{aligned} L^* v^{2q'} = & -2q' v^{2q'-1} \left[\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial v}{\partial x_i} \right) + \sum_{i=1}^n c_i \frac{\partial v}{\partial x_i} - \sum_{i=1}^n \frac{\partial}{\partial x_i} (b_i v) \right] \\ & -2q'(2q'-1) v^{2q'-2} \sum_{i,j=1}^n a_{ij} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} + \left((2q'-1) v^{2q'} \frac{\partial b_i}{\partial x_i} + d \right) v^{2q'}. \end{aligned}$$

Then

$$\int_\Omega \frac{|L^* v^{2q'}|^{q'}}{v^{2q'(q'-1)}} dx \leq C \|v\|_{L^\infty}^{q'} \|v\|_{W^{2,q'}}^{q'} + \|\nabla v\|_{L^{2q'}}^{2q'},$$

and finally

$$\int_\Omega v^{2q'} d\lambda_\sigma \leq C \|v\|_{L^\infty}^{q'} \|v\|_{W^{2,q'}}^{q'}, \tag{4.46}$$

by the Gagliardo-Nirenberg inequality. If $K \subset \Omega$ is compact, there exists a sequence $\{v_k\} \subset C_c^\infty(\Omega)$ such that $0 \leq v_k \leq 1$, $v_k \equiv 1$ in a neighborhood of K and $\|v\|_{k W^{2,q'}}^{q'} \to C_{2,q'}(K)$ when $k \to \infty$. Therefore (4.46) implies (4.44). $\square$

Remark. In the particular case where $K = B_r(x_0)$ (for $0 < r < \rho_{\partial\Omega}(x_0)$, the measure λ_σ satisfies

$$\lambda_\sigma(B_r(x_0)) \leq C \begin{cases} r^{n-2q'} & if \ q > n/(n-2), \\ (\ln(1+1/r))^{1-q'} & if \ q = n/(n-2). \end{cases} \tag{4.47}$$

Estimate (4.44) can be understood in saying that the measure λ_σ is Lipschitz continuous with respect to the capacity $C_{2,q'}$, although it must be noticed that a capacity is only an outer measure, not a regular one.

Later on, Adams and Pierre [2] proved a series of remarquable equivalent properties linking estimates of type (4.44) and Bessel capacities.

Theorem 4.9 *Let $n > 2$, $p > 1$ and λ be a nonnegative measure with compact support in Ω. Then the following conditions are equivalent :*

(i) There exists $k_1 > 0$ such that for all compact subset $K \subset \Omega$,

$$\lambda(K) \leq k_1 C_{2,p}(K). \tag{4.48}$$

(ii) There exists $k_2 > 0$ such that

$$\int_\Omega \xi^p d\lambda \leq k_2 \int_\Omega |\Delta\xi|^p dx, \quad \forall \xi \in Y(-\Delta). \tag{4.49}$$

(iii) There exists $k_3 > 0$ such that

$$\int_\Omega \xi d\lambda \leq k_3 \int_\Omega |\Delta\xi|^p \xi^{1-p} dx, \quad \forall \xi \in Y(-\Delta). \tag{4.50}$$

(iv) There exists $k_4 > 0$ such that

$$\int_\Omega \xi d\lambda \leq k_4 \int_\Omega |L^*\xi|^p \xi^{1-p} dx, \quad \forall \xi \in Y(L^*). \tag{4.51}$$

Their proof is performed with an elliptic operator with C^1 coefficients, but it can be adapted to an operator satisfying condition (H). It heavily relies on fine properties of real valued functions in connection with the Hardy-Littlewood maximal function and the Muckenhoupt weights.

Usually a positive measure $\lambda \in W^{-2,q}(\Omega)$ does not satisfies (4.48), but only

$$\lambda(G) \leq \|\lambda\|_{W^{-2,q}(\Omega)} C^{1/q}_{2,q'}(G), \quad \forall G \subset \Omega,\ G \text{ compact}. \tag{4.52}$$

However, the capacitary measure λ_K of a compact subset of $K \subset \Omega$ does verify it. This measure is the unique extremal for the dual definition of the capacity of K given by (3.54). It is concentrated on K and has the property that

$$\lambda_K(K) = C_{2,q'}(K), \tag{4.53}$$

(see [1, Th 2.2.7]). Moreover

$$G_1 * \lambda_K \in L^q(\mathbb{R}^n) \text{ and } G_1 * (G_1 * \lambda_K)^{q-1} \in L^\infty(\mathbb{R}^n). \tag{4.54}$$

where G_1 denotes the Bessel kernel of order 1 defined by (3.50). The following result is proven in [84].

Proposition 4.10 *Let $K \subset \Omega$ be compact subset with $C_{2,q'}(K) > 0$ and λ_K the capacitary measure of K. Then there exists $k = k(n,q)$ such that*

$$\int_\Omega \xi d\lambda_K \leq k \left\|G_1 * (G_1 * \lambda_K)^{q-1}\right\|_{L^\infty(\mathbb{R}^n)} \int_\Omega |\Delta\xi|^{q'} \xi^{1-q'} dx, \quad \forall \xi \in Y(-\Delta). \tag{4.55}$$

Hence, by Corollary 4.6, Problem 4.35 is solvable for any capacitary measure $\lambda = \lambda_K$, for $0 < \sigma \leq \sigma_0$ for some $\sigma_0 > 0$. Furthermore, since it is proven in [54, Th. 3.1] that there exists a constant $k_{n,q} > 0$ such that

$$\left\|G_1 * (G_1 * \lambda_K)^{q-1}\right\|_{L^\infty(\mathbb{R}^n)} \leq k_{n,q} \quad \forall K \subset \Omega,\ K \text{ compact},$$

it follows that $\sigma_0 = \sigma_0(n,q)$.

4.3 Semilinear equations with power source terms

In this section we develop a direct methods for constructing explicit super solutions in order to apply Theorem 4.2. We assume that Ω is a bounded open subset with a C^2 boundary and that L defined by (2.1) satisfies (H).

Theorem 4.11 *Let $q>0$, $\lambda \in \mathfrak{M}_+(\Omega;\rho_{\partial\Omega})$. If there exists some $C_0>0$ such that*

$$\mathbb{G}_L^\Omega\left(\left(\mathbb{G}_L^\Omega(\lambda)\right)^q\right) \leq C_0\mathbb{G}_L^\Omega(\lambda), \qquad a.e.\ in\ \Omega, \tag{4.56}$$

then problem

$$\begin{aligned} Lu &= |u|^{q-1}u + \sigma\lambda \quad && in\ \ \Omega, \\ u &= 0 && on\ \ \partial\Omega, \end{aligned} \tag{4.57}$$

admits a positive solution $u \in L^1(\Omega)\cap L^q(\Omega;\rho_{\partial\Omega}dx)$,
(i) if $0<\sigma\leq\sigma_0=\sigma_0(q,C_0)$, when $q>1$,
(ii) for any $\sigma>0$ when $0<q\leq 1$.

Proof. Put $w=\theta\mathbb{G}_L^\Omega(\sigma\lambda)$, for some parameters $\theta,\sigma>0>0$. Then, under condition (4.56),

$$\mathbb{G}_L^\Omega(w^q+\sigma\lambda) \leq (C_0\theta^q\sigma^q+\sigma)\mathbb{G}_L^\Omega(\lambda).$$

Therefore

$$w \geq \mathbb{G}_L^\Omega(w^q)+\mathbb{G}_L^\Omega(\sigma\lambda), \tag{4.58}$$

as soon as

$$C_0\theta^q\sigma^{q-1}+1\leq\theta. \tag{4.59}$$

If $q>1$ this is equivalent to

$$\sigma \leq \max_{\theta>0}\left(\frac{\theta-1}{C_0\theta}\right)^{1/(q-1)} = \frac{1}{q(C_0q)^{1/(q-1)}},$$

and we get (i) by Theorem 4.2. If $0<q\leq 1$, for any $\sigma>0$ one can find $\theta>0$ such that (4.59) holds. $\square$

The next result due to [52] ([20] if $L=-\Delta$) points out how close to a necessary condition estimate (4.56) is.

Theorem 4.12 *Let $q>1$, $\lambda\in\mathfrak{M}_+(\Omega;\rho_{\partial\Omega})$, $\sigma>0$. If there is a positive solution $u\in L^1(\Omega)$ to Problem (4.57), there exists a constant $C_1>0$ such that*

$$\mathbb{G}_L^\Omega\left(\left(\mathbb{G}_L^\Omega(\sigma\lambda)\right)^q\right) \leq C_1\mathbb{G}_L^\Omega(\sigma\lambda), \qquad a.e.\ in\ \Omega. \tag{4.60}$$

If $L=-\Delta$, $C_1=1/(q-1)$.

Lemma 4.13 *Let $h \in L^1(\Omega;\rho_{\partial\Omega}dx)$, $h \geq 0$, and $\mu, \eta \in \mathfrak{M}_+(\Omega;\rho_{\partial\Omega})$, $\mu \neq 0$, such that $\mu - \eta \geq h$. If $\phi \in C^2([0,\infty))$ is a concave nondecreasing function such that $\phi(1) \geq 0$, there holds*

$$h\phi'\left(\frac{\mathbb{G}^\Omega_{-\Delta}(\mu)}{\mathbb{G}^\Omega_{-\Delta}(\eta)}\right) \in L^1(\Omega;\rho_{\partial\Omega}dx), \tag{4.61}$$

and

$$-\Delta\left(\phi\left(\frac{\mathbb{G}^\Omega_{-\Delta}(\mu)}{\mathbb{G}^\Omega_{-\Delta}(\eta)}\right)\mathbb{G}^\Omega_{-\Delta}(\eta)\right) \geq h\phi'\left(\frac{\mathbb{G}^\Omega_{-\Delta}(\mu)}{\mathbb{G}^\Omega_{-\Delta}(\eta)}\right). \tag{4.62}$$

Proof. Put $z = \mathbb{G}^\Omega_{-\Delta}(\mu)$ and $w = \mathbb{G}^\Omega_{-\Delta}(\eta)$. We write $\eta = h + \mu + \sigma$ where σ is a positive Radon measure. Let h_n, μ_n and σ_n be elements of $C^\infty_c(\Omega)$ such that $h_n \to h$ in $L^1(\Omega;\rho_{\partial\Omega}dx$, and $\mu_n \to \mu$ and $\sigma_n \to \sigma$, in the weak sense of $\mathfrak{M}_+(\Omega;\rho_{\partial\Omega})$. Put $z_n = \mathbb{G}^\Omega_{-\Delta}(\mu_n)$ and $w_n = \mathbb{G}^\Omega_{-\Delta}(h_n + \mu_n + \sigma_n)$, then $z_n \to z$ and $w_n \to w$ in $L^1(\Omega)$ as $n \to \infty$, and a.e. (after extraction of a subsequence). Thus $z_n > 0$ in Ω, for n large enough. Because of the concavity, $\phi(1) \geq 0$ and $\phi' \geq 0$, there holds

$$-\Delta\left(z_n\phi\left(\frac{w_n}{z_n}\right)\right) \geq \phi'\left(\frac{w_n}{z_n}\right)(h_n + \sigma_n) \geq \phi'\left(\frac{w_n}{z_n}\right)h_n.$$

Also

$$0 \leq z_n\phi\left(\frac{w_n}{z_n}\right) \leq z_n\left(\phi_0 + \phi'(0)\frac{w_n}{z_n}\right) \leq C(z_n + w_n),$$

for some $C > 0$. Therefore $z_n\phi(w_n/z_n)$ converges in $L^1(\Omega)$ as $n \to \infty$. Since for any $\xi \in C^{1,1}_c(\overline{\Omega})$, $\xi \geq 0$, there holds

$$-\int_\Omega z_n\phi\left(\frac{w_n}{z_n}\right)\Delta\xi dx \geq \int_\Omega \phi'\left(\frac{w_n}{z_n}\right)h_n\xi dx, \tag{4.63}$$

we derive (4.62) by passing to the limit with Lebesgue and Fatou's theorems. $\square$

Proof of Theorem 4.12. First, we prove the result when $L = -\Delta$. Since $\sigma > 0$, we can assume $\sigma = 1$ and apply Lemma 4.13 with $w = u$, the solution of (4.57), $z = G^\Omega_{-\Delta}(\lambda)$ and

$$\phi(s) = \begin{cases} (1 - s^{1-q})/(q-1), & \text{if } s \geq 1, \\ s - 1, & \text{if } s \leq 1. \end{cases}$$

Because $u \geq G^\Omega_{-\Delta}(\lambda)$,

$$-\Delta\left(G^\Omega_{-\Delta}(\lambda)\phi\left(\frac{u}{G^\Omega_{-\Delta}(\lambda)}\right)\right) \geq \phi'\left(\frac{u}{G^\Omega_{-\Delta}(\lambda)}\right)u^q = \left(G^\Omega_{-\Delta}(\lambda)\right)^q, \tag{4.64}$$

holds weakly. By the maximum principle,

$$\frac{1}{q-1}G^\Omega_{-\Delta}(\lambda) - \frac{1}{q-1}u^{1-q}\left(G^\Omega_{-\Delta}(\lambda)\right)^q \geq G^\Omega_{-\Delta}\left(\left(G^\Omega_{-\Delta}(\lambda)\right)^q\right), \tag{4.65}$$

which is the expected inequality in the case $L = -\Delta$. We turn now to the general case. By Theorem 2.11, the Green functions of L and $-\Delta$ are equivalent in the sense that

$$C^{-1}G^{\Omega}_{-\Delta}(x,y) \leq G^{\Omega}_{L}(x,y) \leq CG^{\Omega}_{-\Delta}(x,y), \quad \forall (x,y) \in \Omega \times \Omega \setminus D_{\Omega},$$

for some $C > 0$. Thus (4.61) follows. $\square$

Remark. In [52], inequality (4.61) is proven for a very general class of positive kernels, not only for a Green kernel.

The next result, proven in [15], exhibits a large class of measures for which Problem (4.57) will be solvable by applying Theorem 4.11.

Theorem 4.14 *Let* $q > 0$, $\alpha \in [0,1]$ *and* $\lambda \in \mathfrak{M}_+(\Omega; \rho^{\alpha}_{\partial\Omega})$ *with* $\|\lambda\|_{\mathfrak{M}_+(\Omega;\rho^{\alpha}_{\partial\Omega})} = 1$. *If*

$$q < \frac{n+\alpha}{n+\alpha-2}, \tag{4.66}$$

then $\mathbb{G}^{\Omega}_{L}(\lambda) \in L^1(\Omega; \rho^{\alpha}_{\partial\Omega}dx)$, *and there exists a positive constant* $C = C(n,q,\alpha,\lambda,\Omega)$ *such that*

$$\mathbb{G}^{\Omega}_{L}\left(\left(G^{\Omega}_{L}(\lambda)\right)^{q}\right) \leq CG^{\Omega}_{L}(\lambda) \quad a.e. \text{ in } \Omega. \tag{4.67}$$

Proof. As in the proof of Theorem 4.12, it is sufficient to consider the case $L = -\Delta$ and then use the equivalence of Green kernels.

Step 1 The case $\lambda = \delta_y$ for $y \in \Omega$, $n \geq 3$. Since $G^{\Omega}_{-\Delta}(x,y) \leq C(n)\,|x-y|^{2-n}$ we put $d = $diam$(\Omega)$ and

$$h(x) = \begin{cases} |x-y|^{2-(n-2)q} & \text{if } q > 2/(n-2), \\ d - |x-y|^{2-(n-2)q} & \text{if } q < 2/(n-2), \\ \ln(d/|x-y|) & \text{if } q = 2/(n-2). \end{cases} \tag{4.68}$$

Hence

$$-\Delta h(.) = C_1\,|.-y|^{(2-n)q} \quad \text{in } \mathcal{D}'(\Omega),$$

and consequently

$$\mathbb{G}^{\Omega}_{-\Delta}\left(\left(G^{\Omega}_{-\Delta}(.,y)\right)^{q}\right)(x) \leq C_2 h(x) \leq C_3\,|x-y|^{2-n},$$

with $C_i = C_i(n,q,d) > 0$. Let $r > 0$ be such that $\overline{B}_r(y) \subset \Omega$. Clearly

$$\mathbb{G}^{\Omega}_{-\Delta}\left(\left(G^{\Omega}_{-\Delta}(.,y)\right)^{q}\right)(x) \leq C'_y\rho_{\partial\Omega}(x) \leq C''_y G^{\Omega}_{-\Delta}(x,y),$$

on $\overline{B}_r(y) \setminus \{y\}$. On $\Omega \setminus B_r(y)$ the function $\mathbb{G}^{\Omega}_{-\Delta}\left(\left(G^{\Omega}_{-\Delta}(.,y)\right)^{q}\right)$ is C^1. We get a similar inequality by Hopf boundary lemma. Finally there exists $C_y > 0$ such that

$$\mathbb{G}^{\Omega}_{-\Delta}\left(\left(G^{\Omega}_{-\Delta}(.,y)\right)^{q}\right)(x) \leq C_y G^{\Omega}_{-\Delta}(x,y), \quad \forall x \in \Omega \setminus \{y\}. \tag{4.69}$$

As we shall see it in next step, C_y is bounded independently of y.

Step 2 The general case. By Theorem 3.5, $\mathbb{G}^{\Omega}_{-\Delta}(\lambda) \in L^q(\Omega; \rho^{\alpha}_{\partial\Omega} dx)$ since (4.66) holds. First assume $q \geq 1$, then

$$\mathbb{G}^{\Omega}_{-\Delta}(\lambda)(x) = \int_{\Omega} G^{\Omega}_{-\Delta}(x,y) d\lambda(y) = \int_{\Omega} \frac{G^{\Omega}_{-\Delta}(x,y)}{\rho^{\alpha}_{\partial\Omega}(y)} \rho^{\alpha}_{\partial\Omega}(y) d\lambda(y).$$

By Jensen's inequality,

$$\left(\mathbb{G}^{\Omega}_{-\Delta}(\lambda)(x)\right)^q \leq \int_{\Omega} \left(\frac{G^{\Omega}_{-\Delta}(x,y)}{\rho^{\alpha}_{\partial\Omega}(y)}\right)^q \rho^{\alpha}_{\partial\Omega}(y) d\lambda(y),$$

$$\mathbb{G}^{\Omega}_{-\Delta}\left(\left(\mathbb{G}^{\Omega}_{-\Delta}(\lambda)\right)^q\right)(x) \leq \int_{\Omega} \mathbb{G}^{\Omega}_{-\Delta}\left(G^{\Omega}_{-\Delta}(.,y)\right)(x) \rho^{\alpha(1-q)}_{\partial\Omega}(y) d\lambda(y).$$

Now

$$\mathbb{G}^{\Omega}_{-\Delta}\left(G^{\Omega}_{-\Delta}(.,y)\right)(x) \rho^{\alpha(1-q)}_{\partial\Omega}(y) = \int_{\Omega} G^{\Omega}_{-\Delta}(x,z) G^{\Omega}_{-\Delta}(y,z) \left(\frac{G^{\Omega}_{-\Delta}(y,z)}{\rho_{\partial\Omega}(y)}\right)^{q-1} dz.$$

Because

$$G^{\Omega}_{-\Delta}(y,z) \leq C \min\{|y-z|^{2-n}, \rho_{\partial\Omega}(y) |y-z|^{1-n}\}, \tag{4.70}$$

it follows

$$G^{\Omega}_{-\Delta}(y,z) \leq C \rho^{\alpha}_{\partial\Omega}(y) |y-z|^{2-n-\alpha}.$$

At that point of the proof we recall the following relation called the 3-G inequality (see [30] for example),

$$\frac{G^{\Omega}_{-\Delta}(x,z) G^{\Omega}_{-\Delta}(y,z)}{G^{\Omega}_{-\Delta}(x,y)} \leq C \left(|x-z|^{2-n} + |y-z|^{2-n}\right), \tag{4.71}$$

where $C = C(\Omega)$. It implies

$$\mathbb{G}^{\Omega}_{-\Delta}\left(G^{\Omega}_{-\Delta}(.,y)\right)(x) \rho^{\alpha(1-q)}_{\partial\Omega}(y) \leq G^{\Omega}_{-\Delta}(x,y) I(x,y),$$

for some $C = C(q, \Omega, \alpha)$, and

$$I(x,y) = \int_{\Omega} |y-z|^{(2-n-\alpha)(q-1)} \left(|x-z|^{2-n} + |y-z|^{2-n}\right) dz.$$

Since

$$I(x,y) \leq C \int_{\Omega} \left(|x-z|^{2-n+(2-n-\alpha)(q-1)} + |y-z|^{2-n+(2-n-\alpha)(q-1)}\right) dz,$$

this last quantity is clearly bounded independently of x and y by some constant depending on the various parameters and data. Notice that we have used

$$q < (n+\alpha)/(n+\alpha-2) \leq n/(n-2).$$

Thus

$$\mathbb{G}_{-\Delta}^{\Omega}\left(\left(\mathbb{G}_{-\Delta}^{\Omega}(\lambda)\right)^{q}\right)(x) \leq C\int_{Gw} G_{-\Delta}^{\Omega}(x,y)d\lambda(y) = C\mathbb{G}_{-\Delta}^{\Omega}(x). \tag{4.72}$$

Obviously, $C = C(\Omega)$ when $q = 1$.

Next we assume $0 \leq q < 1$. Then

$$\mathbb{G}_{-\Delta}^{\Omega}\left(\left(\mathbb{G}_{-\Delta}^{\Omega}(\lambda)\right)^{q}\right) \leq \mathbb{G}_{-\Delta}^{\Omega}(1) + \mathbb{G}_{-\Delta}^{\Omega}\left(\left(\mathbb{G}_{-\Delta}^{\Omega}(\lambda)\right)\right).$$

By Hopf boundary lemma $\mathbb{G}_{-\Delta}^{\Omega}(1)(x) \leq C\rho_{\partial\Omega}(x)$. Let K be a compact subset contained in the support of λ and denote by $\lambda|_K$ the restriction of λ to K. By the regularity results, $\mathbb{G}_{-\Delta}^{\Omega}(\lambda|_K) \in C^1(\overline{\Omega} \setminus K)$. Then $\mathbb{G}_{-\Delta}^{\Omega}(\lambda) \geq \mathbb{G}_{-\Delta}^{\Omega}(\lambda|_K) \geq C\rho_{\partial\Omega}$ in $\overline{\Omega} \setminus K$. In turn it implies $\mathbb{G}_{-\Delta}^{\Omega}(\lambda) \geq C\rho_{\partial\Omega}$ for another constant $C > 0$ and (4.67) follows. □

Condition (4.66) on q is called α-subcriticality. However, as we have seen it in previous sections, there exists measures for which (4.57) is solvable even if q is not α-subcritical.

Definition 4.15 A measure $\lambda \in \mathfrak{M}_+(\Omega;\rho_{\partial\Omega}^{\alpha})$ is called q-admissible if there exists some $\sigma_0 \geq 0$ such that Problem (4.57) admits a solution $u \in L^1(\Omega) \cap L^q(\Omega;\rho_{\partial\Omega}dx)$ whenever $0 < \sigma \leq \sigma_0$.

The following theorem summarizes the results of Baras and Pierre [10], Adams and Pierre [2] and Kalton and Verbitsky [52] in the super-critical range of exponents.

Theorem 4.16 *Let $q > 1$, $\alpha \in [0,1]$ and $\lambda \in \mathfrak{M}_+(\Omega;\rho_{\partial\Omega}^{\alpha})$. Then the following conditions are equivalent :*

(i) λ is q-admissible.

(ii) There exists some $C_0 > 0$ such that

$$\mathbb{G}_{L}^{\Omega}\left(\left(\mathbb{G}_{L}^{\Omega}(\lambda)\right)^{q}\right) \leq C_0\mathbb{G}_{L}^{\Omega}(\lambda). \tag{4.73}$$

(iii) $\left(\mathbb{G}_{L}^{\Omega}(\lambda)\right)^{q}$ is q-admissible.

(iv) There exists $C > 0$ such that

$$\int_{\Omega}\mathbb{G}_{L}^{\Omega}(\lambda)dx \leq C\int_{\Omega}\frac{g^{q'}}{\left(\mathbb{G}_{L}^{\Omega}(g)\right)^{q'-1}}dx, \quad \forall g \in L_c^{\infty}(\Omega),\ g \geq 0. \tag{4.74}$$

(iv) There exists $c > 0$ such that

$$\int_{A} d\lambda \leq cC_{2,q',\alpha}(A), \quad \forall A \subset \Omega,\ A \text{ Borel}, \tag{4.75}$$

where $C_{2,q',\alpha}$ is the weighted capacity defined by

$$C_{2,q',\alpha}(A) = \inf\left\{\int_{\Omega}\eta^{q'}dx : \eta \in L^{q'}(\Omega),\ \eta \geq 0,\ \mathbb{G}_{L^*}^{\Omega}(\lambda) \geq \rho_{\partial\Omega}^{\alpha} \text{ on } A\right\}. \tag{4.76}$$

4.4 Isolated singularities

If one looks for radial positive solutions of

$$-\Delta u = |u|^{q-1} u, \tag{4.77}$$

with $q > 1$, in $\mathbb{R}^n \setminus \{0\}$ under the form $x \mapsto a\,|x|^b$, one immediately finds

$$u(x) = u_s(x) = \gamma_{q,n}\,|x|^{-2/(q-1)}, \tag{4.78}$$

where

$$\gamma_{q,n} = \left(\left(\frac{2}{q-1}\right)\left(n - \frac{2q}{q-1}\right)\right)^{1/(q-1)}. \tag{4.79}$$

However such a solution exists if and only if $q > n/(n-2)$. Moreover, if $q \geq n/(n-2)$, it follows by Theorem 3.23 that, if Ω is an open subset of $\mathbb{R}^n$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, and if $u \in L^q_{loc}(\Omega^*)$ is nonnegative and satisfies

$$-\Delta u = u^q \quad \text{in } \mathcal{D}'(\Omega^*), \tag{4.80}$$

then $u \in L^q_{loc}(\Omega)$, and that Equation (4.80) holds in $\mathcal{D}'(\Omega)$. In this way, the singularity of u at 0 exists, but is not visible in the sense of distributions. In the subcritical range, $1 < q < n/(n-2)$ it is proven by Brezis and Lions [21] that any positive solution of (4.80) satisfies actually

$$-\Delta u = u^q + C_n \gamma \delta_0 \quad \text{in } \mathcal{D}(\Omega), \tag{4.81}$$

for some $\gamma \geq 0$ (see Step 4 in the proof of Theorem 3.40). Furthermore u admits an expansion near 0;

$$u(x) = \gamma\,|x|^{2-n}(1 + \circ(1)) + C, \quad \text{as } x \to 0, \tag{4.82}$$

if $n \geq 3$, with the usual modification if $n = 2$. Finally, although this was noticed before by Lions [66], Theorem 4.14 implies that the Dirac mass δ_0 is q-admissible. The classification of isolated singularities of positive solutions of (4.77) has been performed by Lions [66] in the case $1 < q < n/(n-2)$, Aviles [6] in the case $q = n/(n-2)$, Gidas and Spruck [46] when $n/(n-2) < q < (n+2)/(n-2)$ and Caffarelli, Gidas and Spruck [24] in the case $q = (n+2)/(n-2)$. The case $q > (n+2)/(n-2)$ remains essentially open, except if the solutions are supposed to be radial.

Theorem 4.17 *Let Ω be an open subset of $\mathbb{R}^n$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, $q > 0$ and $u \in C^2(\Omega^*)$ be a positive solution of (4.77) in Ω^*.*

(i) If $q < n/(n-2)$: either $u \in C^\infty(\Omega)$, or there exists $\gamma > 0$ such that (4.82) and (4.81) hold.

(ii) If $q = n/(n-2)$: either $u \in C^\infty(\Omega)$, or

$$\lim_{x \to 0} |x|^{n-2} (\ln(1/|x|))^{(2-n)/2} u(x) = \left(\frac{n-2}{\sqrt{2}}\right)^{n-2}. \tag{4.83}$$

(iii) If $n/(n-2) < q < (n+2)/(n-2)$: either $u \in C^\infty(\Omega)$, or

$$\lim_{x \to 0} |x|^{2/(q-1)} u(x) = \gamma_{q,n}. \tag{4.84}$$

(iv) If $q = (n+2)/(n-2)$: either $u \in C^\infty(\Omega)$, or

$$\lim_{x \to 0} |x|^{(n-2)/2} (u(x) - v(|x|) = 0, \tag{4.85}$$

where $r \mapsto v(r)$ is a radial solution of (4.77).

Notice that in the so-called conformal case $q = (n+2)/(n-2)$, all the radial solutions v of (4.77) are classified by their reduced energy : if $v(r) = r^{(2-n)/2} w(t)$ and $t = \ln(1/r)$, then w verifies

$$w'' - \frac{(n-2)^2}{4} w + |w|^{(4)/(n-2)} w = 0. \tag{4.86}$$

Therefore the reduced energy-function

$$\mathcal{E}(w) = w'^2 + \frac{n+2}{n} |w|^{2n/(n+2)} - \frac{(n-2)^2}{4} w^2$$

is constant. The proofs of these different results relies on regularity estimates and boot-strap arguments in case (i), the Lyapounov analysis as for Theorem 3.28 in cases (ii) and (iii), and the asymptotic symmetry method in the case (iv). However, there are two difficulties in case (iii) ((ii) being much simpler) : the first one is to prove the *a priori* estimate

$$u(x) \leq C\,|x|^{2/(q-1)} \quad \text{near } 0. \tag{4.87}$$

The second one is to identify the limit set at the end of the Lyapounov analysis, in which situation, it is to be proven that the only positive solutions to

$$-\Delta_{S^{n-1}} \omega + \gamma_{q,n}^{q-1} \omega - w^q = 0 \tag{4.88}$$

on S^{n-1} are the constant solutions 0 and $\gamma_{q,n}$.

Remark. Part of the results can be extended to equation

$$Lu = u^q, \tag{4.89}$$

where L is a general elliptic operator, satisfying condition (H). This extension is easy for (i), a little more complicated in case (iii) (and (ii) in the same way), in particular to get (4.87). It is still completely open in case (iv).

5 Boundary singularities and boundary trace

In this chapter we shall study generalized boundary value problems for equation

$$Lu + g(x,u) = 0 \quad \text{in } \Omega, \tag{5.1}$$

where Ω is an open domain in $\mathbb{R}^n$, $n \geq 2$, with a C^2 boundary, L is an elliptic operator defined in Ω by (2.1) and g a continuous function of absorption type.

5.1 Measures boundary data

5.1.1 General solvability

Let μ be a Radon measure on $\partial\Omega$ and $g \in C(\Omega \times \mathbb{R}$. The semilinear Dirichlet problem with measure data is written under the form

$$\begin{aligned} Lu + g(x,u) &= 0 \quad \text{in } \Omega, \\ u &= \mu \quad \text{on } \partial\Omega. \end{aligned} \tag{5.2}$$

Definition 5.1 *Let $\mu \in \mathfrak{M}(\partial\Omega)$. A function u is a solution of (5.2), if $u \in L^1(\Omega)$, $g(.,u) \in L^1(\Omega;\rho_{\partial\Omega}dx)$, and if for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, there holds*

$$\int_\Omega (uL^*\zeta + g(x,u)\zeta)\,dx = -\int_\Omega \frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}d\mu. \tag{5.3}$$

Definition 5.2 A real valued function $g \in C(\Omega \times \mathbb{R})$ holds the *boundary-weak-singularity assumption*, if there exists $r_0 \geq 0$ such that

$$rg(x,r) \geq 0, \quad \forall (x,r) \in \Omega \times (-\infty,-r_0] \cup [r_0,\infty), \tag{5.4}$$

and a nondecreasing function $\tilde{g} \in C([0,\infty))$ such that $\tilde{g} \geq 0$,

$$\int_0^1 \tilde{g}(r^{1-n})r^n dr < \infty, \tag{5.5}$$

and

$$|g(x,r)| \leq \tilde{g}(|r|), \quad \forall (x,r) \in \Omega \times \mathbb{R}. \tag{5.6}$$

The following result was proven first, but under a weaker form, by Gmira and Véron [48].

Theorem 5.3 *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 2$, L the elliptic operator defined by (2.1) and $g \in C(\Omega \times \mathbb{R})$ a real valued function. If L satisfies assumptions (H) and g the boundary-weak-singularity assumption, for any $\mu \in \mathfrak{M}(\partial\Omega)$ there exists a solution u to Problem (5.2).*

Proof. The general idea follows the proof of Theorem 3.7, with some significant changes.
Step 1 Approximate solutions. Let μ_n be a sequence of $C^2(\Omega)$ functions converging to μ in the weak sense of measures and $m_n = \mathbb{P}_L^\Omega(\mu_n)$. The function g^n defined by

$$g^n(x,r) = g(x, r - m_n(x)), \quad \forall (x,r) \in \Omega \times \mathbb{R},$$

is continuous in $\Omega \times \mathbb{R}$ and satisfies (5.4) with r_0 replaced by $r_0 + \|m_n\|_{L^\infty}$. By Theorem 3.7 there exists a solution to

$$\begin{aligned} Lv_n + g^n(x,v_n) &= 0 \quad \text{in } \Omega, \\ v_n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.7}$$

Thus the function $u_n = v_n + m_n$ is a solution of

$$\begin{aligned} Lu_n + g(x,u_n) &= 0 \quad \text{ in } \ \Omega, \\ u_n &= \mu_n \quad \text{on } \ \partial\Omega. \end{aligned} \tag{5.8}$$

From the proof of Theorem 3.7, Steps 2-3, u_n is bounded in Ω and (5.3) holds with u_n and m_n. By Theorem 2.4, for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, $\zeta \geq 0$,

$$\int_\Omega (|u_n|\, L^*\zeta + \mathrm{sign(u_n)} g(x,u_n)\zeta)\, dx \leq -\int_\Omega \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} |\mu_n|\, dx, \tag{5.9}$$

which implies

$$\|u_n\|_{L^1(\Omega)} + \|\rho_{\partial\Omega} g(.,u_n)\|_{L^1(\Omega)} \leq \Theta \int_\Omega \rho_{\partial\Omega} dx + C_1 \|\rho_{\partial\Omega}\mu_n\|_{L^1(\partial\Omega)}\,. \tag{5.10}$$

Consequently, using also (3.11) in Theorem 3.5,

$$\|u_n\|_{M^{(n+\alpha)/(n+\alpha-2)}(\Omega;\rho^\alpha_{\partial\Omega})} \leq C_2 \|\lambda_n - g(.,u_n)\|_{\mathfrak{M}(\Omega;\rho^\alpha_{\partial\Omega})} \leq C_3 \left(\Theta + \|\rho_{\partial\Omega}\mu_n\|_{L^1(\partial\Omega)}\right), \tag{5.11}$$

for $\alpha = 0,\, 1$.

Step 2 Convergence. By Corollary 2.8 and (5.11), there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ for simplicity, which converges to some u in $L^1(\Omega)$ and a.e. in Ω. In order to prove that $g(.,u_{n)}$ converges in $L^1(\Omega;\rho_{\partial\Omega}dx)$, we use Vitali's theorem and we procede as in the proof of Theorem 3.7- Step 3 with $\alpha = 1$. $\square$

The following stability result follows from the uniform integrability argument.

Corollary 5.4 *Let g satisfy the boundary-weak-singularity assumption and $r \mapsto g(x,r)$ is nondecreasing, for any $x \in \Omega$. Then the solution u is unique. If we assume that $\{\mu_k\}$ is a sequence of measures in $\mathfrak{M}(\Omega)$ which converges weakly to μ, then the corresponding solutions u_{μ_k} of problem*

$$\begin{aligned} Lu_{\mu_k} + g(x,u_{\mu_k}) &= 0 \quad \textit{in } \ \Omega, \\ u_{\mu_k} &= \mu_k \quad \textit{on } \ \partial\Omega, \end{aligned} \tag{5.12}$$

converge in $L^1(\Omega)$ to the solution u of (5.2), when $k \to \infty$.

Remark. If $g(x,r) = |r|^{q-1}\, r$, the boundary-weak-singularity assumption is satisfied if and only if

$$0 < q < \frac{n+1}{n-1}. \tag{5.13}$$

5.1.2 Admissible boundary measures and the Δ_2-condition

Definition 5.5 Let $\tilde{g}$ be a continuous real valued nondecreasing function defined in $\mathbb{R}_+$, $\tilde{g} \geq 0$. A Radon measure μ in $\partial\Omega$ is called $(\tilde{g}, k)$-boundary-admissible if

$$\int_\Omega \tilde{g}(\mathbb{P}_L^\Omega(|\mu|) + k)\rho_{\partial\Omega} dx < \infty, \tag{5.14}$$

where $P_L^\Omega(|\mu|)$ is the Poisson potential of μ and $k \geq 0$.

The proof of the following theorem is similar to the one of Theorem 3.10.

Theorem 5.6 *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $n \geq 2$, L an elliptic operator defined by (2.1) verifying condition (H), and $g \in C(\Omega \times \mathbb{R})$ satisfying (5.4) for some $r_0 \geq 0$ and (5.6) for some function $\tilde{g}$ as in Definition 3.9. Then for any $(\tilde{g}, r_0)$-boundary-admissible Radon measure $\mu \in \mathfrak{M}(\partial\Omega)$, Problem (5.2) admits a solution.*

The proof of the next result, is a boundary adaptation of the one of Theorem 3.12.

Theorem 5.7 *Let Ω and L be as in Theorem 5.6. Assume $g \in C(\Omega \times \mathbb{R})$ satisfies the Δ_2-condition (3.37), $r \mapsto g(x, r)$ is nondecreasing for any $x \in \Omega$ and (5.6) holds for some nonnegative, nondecreasing function $\tilde{g}$. For any Radon measure $\lambda \in \mathfrak{M}(\partial\Omega)$, with $\lambda = \tilde{\lambda} + \lambda^*$, where $\tilde{\lambda} \in L^1(\partial\Omega)$ and λ^* is $(\tilde{g}, 0)$-boundary-admissible and singular with respect to the $(n-1)$-dimensional Hausdorff measure, problem (5.2) admits a unique solution.*

5.1.3 Sharp solvability

The existence of a solution, necessarily unique, to

$$\begin{aligned} Lu + |u|^{q-1} u = 0 \quad \text{in} \;\; \Omega, \\ u = \mu \quad \text{on} \;\; \partial\Omega, \end{aligned} \tag{5.15}$$

where μ is a boundary measure follows unconditionaly from Theorem 5.3 in the subcritical range $0 < q < (n+1)/(n-1)$. The super-critical case $q \geq (n+1)/(n-1)$ is treated separately according the value of q with respect to 2 by Le Gall [63], Dynkin and Kuznestov [38], [39] and Marcus and Véron [71]. The synthetic presentation in all the super-critical cases is found in [72].

Theorem 5.8 *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2 boundary, L the elliptic operator defined by (2.1) satisfying condition (H), $q \geq (n+1)/(n-1)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. Then Problem (5.15) admits a solution $u = u_\mu$ if and only if μ does not charge boundary sets with $C_{2/q,q'}$-capacity zero. Moreover, the mapping $\mu \mapsto u_\mu$ is increasing.*

Following Definition 5.5, a Radon measure μ on $\partial\Omega$ is called *boundary-q-admissible* for the operator L if

$$\int_\Omega \left(\mathbb{P}_L^\Omega(|\mu|)\right)^q \rho_{\partial\Omega} dx < \infty. \tag{5.16}$$

However, under assumption (H), under which the Green and Poison kernels are constructed, this property is independent of L, since all the kernels are equivalent (see Theorem 2.11). The proof is based upon a deep result concerning representation of boundary Bessel classes in terms of integrability properties of Poisson potentials.

Proposition 5.9 *Let the assumtions of Theorem 5.8, on Ω and the operator L, be satisfied, $q \geq (n+1)/(n-1)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. Then :*
(i) If μ is boundary-q-admissible, then $\mu \in W^{-2/q,q}(\partial\Omega)$.
(ii) If $\mu \in \mathfrak{M}_+(\partial\Omega) \cap W^{-2/q,q}(\partial\Omega)$, then μ is boundary-q-admissible. Moreover there exists a constant $C = C(q, \Omega, L)$ such that,

$$C^{-1}\|\mu\|_{W^{-2/q,q}(\partial\Omega)} \leq \|\mathbb{P}_L^{\Omega}(\mu)\|_{L^q(\Omega;\rho_{\partial\Omega}dx)} \leq C\|\mu\|_{W^{-2/q,q}(\partial\Omega)}. \tag{5.17}$$

Proof. The proof we present here is settled upon the interpolation theory between a Banach space and the domain of an analytic semigroup of operators.
Step 1 The case where Ω is the unit ball B. We shall assume $n \geq 3$, the 2-dimensional case requiring some easy technical modifications. Let (r, σ) be the spherical coordinates in $\mathbb{R}^n$, $t = -\ln r$. If $\mu \in W^{-2/q,q}(S^{n-1})$, we set $u = \mathbb{P}_{-\Delta}^{\Omega}(\mu)$, and $\tilde{u}(t, \sigma) = u(r, \sigma)$. Then relation (5.17) turns into

$$C^{-1}\|\mu\|_{W^{-2/q,q}(S^{n-1})} \leq \int_0^\infty \int_{S^{n-1}} |\tilde{u}|^q (1 - e^{-t})e^{-nt} d\sigma\, dt \leq C\|\mu\|_{W^{-2/q,q}(S^{n-1})}. \tag{5.18}$$

By density it can be assumed that μ is a regular function, and let f be the solution of

$$\mu = \frac{(n-2)^2}{4} f - \Delta_{S^{n-1}} f \quad \text{in} \quad S^{n-1}.$$

By elliptic equations regularity theory, there exists $c > 0$ such that

$$c^{-1}\|\mu\|_{W^{-2/q,q}(S^{n-1})} \leq \|f\|_{W^{2-2/q,q}(S^{n-1})} \leq c\|\mu\|_{W^{-2/q,q}(S^{n-1})}. \tag{5.19}$$

Let $v = \mathbb{P}_{-\Delta}^{\Omega}(f)$ in B and $\tilde{v}(t, \sigma) = v(r, \sigma)$. Then

$$\begin{aligned} \tilde{L}\tilde{v} := \tilde{v}_{tt} - (N-2)\tilde{v}_t + \Delta_{S^{n-1}}\tilde{v} = 0 \quad &\text{in } \mathbb{R}_+ \times S^{n-1}, \\ \tilde{v}|_{t=0} = f \quad &\text{on } S^{n-1}. \end{aligned} \tag{5.20}$$

This implies

$$\tilde{L}(\Delta_{S^{n-1}} s\tilde{v}) = 0 \quad \text{in } \mathbb{R}_+ \times S^{n-1}, \quad \text{and } \Delta_{S^{n-1}}\tilde{v}|_{t=0} = \Delta_{S^{n-1}} f \quad \text{on } S^{n-1}. \tag{5.21}$$

This problem has a unique solution which is bounded near $t = \infty$, therefore

$$\mathbb{P}_{-\Delta}^{\Omega}(\Delta_{S^{n-1}} f) = \Delta_{S^{n-1}}\tilde{v}, \tag{5.22}$$

and equivalently

$$\tilde{u} = \mathbb{P}_{-\Delta}^{\Omega}(\mu) = \mathbb{P}_{-\Delta}^{\Omega}\left(\frac{(n-2)^2}{4} f - \Delta_{S^{n-1}} f\right) = \frac{(n-2)^2}{4}\tilde{v} - \Delta_{S^{n-1}}\tilde{v}. \tag{5.23}$$

Put $v^* := e^{-t(N-2)/2}\tilde{v}$, then

$$\begin{aligned} v^*_{tt} - \frac{(n-2)^2}{4} v^* + \Delta_{_{S^{n-1}}} v^* = 0 \quad & \text{in } \mathbb{R}_+ \times S^{n-1}, \\ v^*(0,\cdot) = f \quad & \text{on } S^{n-1}. \end{aligned} \tag{5.24}$$

One way to represent v^* is to introduce semigroups of linear operators and to express the above relations in terms of interpolation spaces between Banach spaces. Put

$$v^* = e^{tA}(f) \quad \text{where} \quad A = -\left(\frac{(n-2)^2}{4} I - \Delta_{_{S^{n-1}}}\right)^{1/2}.$$

It is wellknown that the square root of a densily defined closed operator A defines an analytic semi-group in $L^q(S^{n-1})$ (see [103] for example). The domain of A^2 is precisely $W^{2,q}(S^{n-1})$. Therefore (see [93, p. 96]),

$$\begin{aligned} \|f\|^q_{W^{2-2/q,q}(S^{n-1})} &\approx \|f\|^q_{L^q(S^{n-1})} + \int_0^\infty \left(t^{2/q} \left\|A^2 v^*\right\|_{L^q(S^{n-1})}\right)^q \frac{dt}{t} \\ &\approx \|f\|^q_{L^q(S^{n-1})} + \int_0^1 \left(t^{2/q} \left\|A^2 v^*\right\|_{L^q(S^{n-1})}\right)^q \frac{dt}{t} \\ &= \|f\|^q_{L^q(S^{n-1})} + \int_0^1 \left(t^{2/q} e^{-t(N-2)/2} \left\|A^2 \tilde{v}\right\|_{L^q(S^{n-1})}\right)^q \frac{dt}{t}, \end{aligned} \tag{5.25}$$

where the symbol $\approx$ denotes equivalence of norms. Notice that for $q > 1$ the exponent $2 - 2/q$ is an integer only if $q = 2$, in which case the Besov and Sobolev spaces coincide. Thus, by (5.19),

$$\begin{aligned} \|f\|^q_{W^{2-2/q,q}(S^{n-1})} &\geq C\|f\|^q_{L^q(S^{n-1})} + C\int_0^1 \left(t^{2/q} e^{-t(n-2)/2} \|\tilde{u}\|_{L^q(S^{n-1})}\right)^q \frac{dt}{t} \\ &\geq C\|f\|^q_{L^q(S^{n-1})} + C\int_0^1 \|\tilde{u}\|^q_{L^q(S^{n-1})} e^{-nt} t\,dt. \end{aligned} \tag{5.26}$$

Since u is an harmonic function,

$$r \mapsto r^{1-n} \int_{\partial B_r} |u|^q dS$$

is nonincreasing on $(0,1]$. Equivalently

$$t \mapsto \int_{S^{n-1}} |\tilde{u}(t,.)|^q d\sigma$$

is nonincreasing on $[0,\infty)$. Furthermore

$$\begin{aligned} \int_0^\infty \|\tilde{u}\|^q_{L^q(S^{n-1})} (1-e^{-t}) e^{-nt} dt &\leq C\int_0^1 \|\tilde{u}\|^q_{L^q(S^{n-1})} (1-e^{-t}) e^{-nt} dt \\ &\leq C\int_0^1 \|\tilde{u}\|^q_{L^q(S^{n-1})} e^{-nt} t\,dt. \end{aligned} \tag{5.27}$$

This inequality implies that

$$\int_{|x|<1} |u|^q(1-r)\,dx \leq c(\gamma)\int_{\gamma<|x|<1} |u|^q(1-r)\,dx,$$

for every $\gamma \in (0,1)$. Because of (5.19),

$$\|\mu\|^q_{W^{-2/q,q}(S^{n-1})} \approx \|f\|^q_{W^{2-2/q,q}(S^{n-1})}. \tag{5.28}$$

Therefore, the right-hand side inequality in (5.17) follows from (5.18), (5.26) and (5.27).

Next assume that μ is a distribution on S^{n-1} and $\mathbb{P}(\mu) \in L^q(B;(1-r)\,dx)$. In order to prove that $\mu \in W^{-2/q,q}(S^{n-1})$ and that the left-hand side inequality in (5.17) holds, we can assume that $\mu \in \mathfrak{M}(S^{n-1})$. By (5.19), if $f \in L^q(S^{n-1})$ then $\mu \in W^{-2/q,q}(S^{n-1})$. Therefore, if it is proven

$$\|f\|_{L^q(S^{n-1})} \leq C\,\|u\|_{L^q(B;(1-r)\,dx)}, \tag{5.29}$$

the left-hand side inequality in (5.17) follows. Equation (5.23) implies that

$$\|v(r,\cdot)\|_{W^{2,q}(S^{n-1})} \leq C\,\|u(r,\cdot)\|_{L^q(S^{n-1})}, \quad \forall r \in (0,1). \tag{5.30}$$

for some $C = C(n) > 0$. Hence

$$\|v\|_{L^q(B;(1-r)\,dx)} + \left\|\Delta_{S^{n-1}} v\right\|_{L^q(B;(1-r)\,dx)} \leq C\,\|u\|_{L^q(B;(1-r)\,dx)}. \tag{5.31}$$

We write (5.20) under the form

$$\begin{cases} \tilde{v}_{tt} - (N-2)\tilde{v}_t = \tilde{h} := -\Delta_{S^{n-1}}\tilde{v} & \text{in } \mathbb{R}_+ \times S^{n-1}, \\ \tilde{v}|_{t=0} = f, & \text{in } S^{n-1}. \end{cases} \tag{5.32}$$

Since $u \in L^q(B;(1-r)\,dx)$, (5.30) implies that $h \in L^q(B;(1-r)\,dx)$ (where $h(x) = \tilde{h}(t,\sigma)$). Let σ be a fixed but arbitrary point on S^{n-1}. Since Equation (5.32) is a first order o.d.e. in $\tilde{v}_t(\cdot,\sigma)$ with a forcing term $\tilde{h}(.,\sigma)$, we fix some initial time $t_0 \in (0,\infty)$ and compute the value of the solution in $(0,t_0)$. Integrating twice one derives

$$\begin{aligned} \tilde{v}(t,\sigma) = \int_{t_0}^{t} e^{(N-2)s}\int_{t_0}^{s} e^{-(N-2)\tau}\tilde{h}(\tau,\sigma)\,d\tau\,ds \\ + \frac{1}{N-2}(e^{(N-2)(t-t_0)}-1)\tilde{v}_t(t_0,\sigma) + \tilde{v}(t_0,\sigma). \end{aligned} \tag{5.33}$$

Therefore

$$\begin{aligned} |v(0,\sigma)| = |f(\sigma)| &\leq C\Big(\int_0^{t_0}\int_s^{t_0}\left|\tilde{h}(\tau,\sigma)\right|\,d\tau\,ds + |\tilde{v}_t(t_0,\sigma)| + |\tilde{v}(t_0,\sigma)|\Big) \\ &= C\Big(\int_0^{t_0} s\left|\tilde{h}(s,\sigma)\right|\,ds + |\tilde{v}_t(t_0,\sigma)| + |\tilde{v}(t_0,\sigma)|\Big) \\ &\leq C\Big(\int_{e^{-t_0}}^{1}(1-r)\,|h(r,\sigma)|\,r^{N-1}dr + |\tilde{v}_t(t_0,\sigma)| + |\tilde{v}(t_0,\sigma)|\Big), \end{aligned} \tag{5.34}$$

where C is a constant independent of t_0, for $t_0 \leq \ln 2$. Taking the q-power and integrating over S^{n-1} yields to

$$\int_{S^{n-1}} |f|^q \, d\sigma \leq C\Big(\int_{r_0<|x|<1} |h|^q \,(x)(1-|x|)\, dx \\ + \int_{S^{n-1}} |v_r|^q \,(r_0,\sigma)\, d\sigma + \int_{S^{n-1}} |v|^q \,(r_0,\sigma)\, d\sigma \Big),$$

where C is independent of r_0, for $r_0 \geq 1/2$. We multiply the inequality by r_0^{N-1} and integrate with respect to r_0 in $(5/8, 6/8)$. It follows that

$$\int_{S^{n-1}} |f|^q \, d\sigma \leq C\Big(\int_{1/2<|x|<1} |h|^q \,(x)(1-|x|)\, dx \\ + \int_{5/8<|x|<6/8} |v_r|^q \, dx + \int_{5/8<|x|<6/8} |v|^q \, dx \Big). \tag{5.35}$$

By interior elliptic estimates,

$$\int_{5/8<|x|<6/8} |v_r|^q \, dx \leq \int_{1/2<|x|<7/8} |v|^q \, dx. \tag{5.36}$$

Finally, by (5.35), (5.36) and (5.31) we obtain (5.29).

Step 2 The case of a general operator L in B. Because of the equivalence property of Theorem 2.11 already mentioned, if $\mu \geq 0$, there exists a constant C such that, for every measure $\mu \in \mathfrak{M}_+(S^{N-1})$,

$$C^{-1}\mathbb{P}^{\Omega}_{-\Delta}(\mu) \leq \mathbb{P}^{\Omega}_{L}(\mu) \leq C\mathbb{P}^{\Omega}_{-\Delta}(\mu). \tag{5.37}$$

Therefore, if (5.17) holds with respect to $\mathbb{P}^{\Omega}_{-\Delta}$, it holds for $\mathbb{P}^{\Omega}_{L}$, for every measure $\mu \in W^{-2/q,q}(S^{n-1}) \cap \mathfrak{M}_+(S^{n-1})$. If μ is a boundary-q-admissible measure for L, not necessarily positive, then μ_+ and μ_- are boundary-q-admissible. Therefore μ_+, $\mu_- \in W^{-2/q,q}(S^{N-1})$, and the same holds with μ. Furthermore

$$C^{-1}\|\mu_\pm\|_{W^{-2/q,q}(\partial\Omega)} \leq \|\mathbb{P}^{\Omega}_{L}(\mu_\pm)\|_{L^q(\Omega;\rho_{\partial\Omega}dx)} \leq C\|\mu_\pm\|_{W^{-2/q,q}(\partial\Omega)}. \tag{5.38}$$

Step 3 The case of a general operator L in a general bounded C^2 domain Ω. There exists a finite set of bounded open subdomains U_i $(1 \leq i \leq k)$ of $\mathbb{R}^n$ such that

$$\partial\Omega \subset \bigcup_{i=1}^{k} U_i,$$

and for each i there exists a C^2 diffeomorphism Φ_i from U_i of into some open subset V_i such that $\Phi_i(U_i \cap \Omega) = B$, and $\Phi_i(U_i \cap \partial\Omega) = \Gamma_i \subset \partial B \approx S^{n-1}$. This diffeomorphism induces an isomorphism, say Φ_i^*, between $\mathfrak{M}(U_i \cap \partial\Omega)$ and $\mathfrak{M}(\Gamma_i)$, $W^{-2/q,q}(U_i \cap \partial\Omega)$ and $W^{-2/q,q}(\Gamma_i)$, and it preserves positivity. Moreover, by the change of variables $x \in U_i \mapsto y = \Phi_i(x) \in V_i$, the operator L is transformed into an elliptic operator L_i^* on B, which still satisfies the maximum principle, not necessarily the condition (2.5), but this is not crucial for the equivalence property in small domains. If $\mu \in \mathfrak{M}(\partial\Omega)$ has its support in $U_i \cap \partial\Omega$, the function $u = \mathbb{P}^L_\Omega(\mu)$ satisfies

$$\begin{aligned} Lu &= 0 && \text{in } U_i \cap \Omega, \\ u &= \mu && \text{on } U_i \cap \partial\Omega, \\ u &= u_c && \text{on } \partial U_i \cap \Omega, \end{aligned} \tag{5.39}$$

where u_c, the restriction of u to $\partial U_i \cap \Omega$, is C^1. Thus the function $v_i = u \circ \Phi_i^{-1}$ satisfies

$$\begin{aligned} L_i^* v_i &= 0 && \text{in } B, \\ v_i &= \Phi_i^*(\mu) && \text{on } \Gamma_i, \\ v_i &= u_c \circ \Phi_i^{-1} && \text{on } \partial B \setminus \Gamma_i. \end{aligned} \tag{5.40}$$

Therefore, if μ is nonnegative and $\mathbb{P}^\Omega_L(\mu) \in L^q(\Omega; \rho_{\partial\Omega} dx)$, $v_i \in L^q(B; (1-|y|)dy)$, which leads to $\Phi_i^*(\mu) \in W^{-2/q,q}(\Gamma_i)$ and $\mu \in W^{-2/q,q}(\partial\Omega)$. Moreover

$$\begin{aligned} \|\mu\|_{W^{-2/q,q}(\partial\Omega)} \approx \|\Phi_i^*(\mu)\|_{W^{-2/q,q}(S^{n-1})} &\leq C \|v_i\|_{L^q(B;(1-|x|)dx)} \\ &\approx \|u\|_{L^q(\Omega\cap U_i;\rho_{\partial(\Omega\cap U_i)} dx)} . \end{aligned} \tag{5.41}$$

Since $\rho_{\partial(\Omega\cap U_i)} \leq \rho_{\partial\Omega}$ in $U_i \cap \Omega$, the integral term on the right in (5.41) is dominated by the norm of $\mathbb{P}^\Omega_L(\mu)$ in $L^q(\Omega; \rho_{\partial\Omega} dx)$. By using a partition of unity, any measure μ on $\partial\Omega$ can be decomposed in the sum of measures μ_i with compact support in Γ_i. Hence the following estimate holds when $P^L_\Omega(\mu) \in L^q(\Omega; \rho_{\partial\Omega} dx)$:

$$\|\mu\|_{W^{-2/q,q}(\partial\Omega)} \leq C \left\|\mathbb{P}^\Omega_L(\mu)\right\|_{L^q(\Omega;\rho_{\partial\Omega} dx)} . \tag{5.42}$$

Conversely, if we assume that $\mu \in \mathfrak{M}_+(\partial\Omega) \cap W^{-2/q,q}(\partial\Omega)$ with support in some fixed compact $K_i \subset \partial\Omega \cap U_i$, then $\Phi_i^*(\mu) \in \mathfrak{M}_+(S^{n-1}) \cap W^{-2/q,q}(S^{n-1})$ with support in Γ_i and equivalence of norms. Then $\mathbb{P}^B_{L^*}(\Phi_i^*(\mu)) \in L^q(B; (1-|x|)dx)$, with

$$\left\|\mathbb{P}^B_{L^*}(\Phi_i^*(\mu))\right\|_{L^q(B;(1-|x|)dx)} \leq C \|\Phi_i^*(\mu)\|_{W^{-2/q,q}(S^{n-1})} \approx \|\mu\|_{W^{-2/q,q}(\partial\Omega)} . \tag{5.43}$$

But the left-hand side term in (5.43) is comparable to $\left\|\mathbb{P}^{U_i\cap\Omega}_L(\mu)\right\|_{L^q(\Omega\cap U_i;\rho_{\partial(\Omega\cap U_i)} dx)}$, and

$$\left\|\mathbb{P}^B_{L^*}(u_c \circ \Phi_i^{-1} \chi_{\partial B\setminus\Gamma_i})\right\|_{L^q(B)} \approx \left\|\mathbb{P}^{U_i\cap\Omega}_L(u_c)\right\|_{L^q(U_i\cap\Omega)} . \tag{5.44}$$

Because u is an harmonic function,

$$\|u_c\|_{L^\infty(\partial U_i\cap\Omega)} \leq C \|\mu\|_{W^{-2/q,q}(\partial\Omega)} . \tag{5.45}$$

Finally

$$u = \mathbb{P}^\Omega_L(\mu) = \begin{cases} \mathbb{P}^{\Omega\setminus U_i}_L(\mu) + \mathbb{P}^{\Omega\setminus U_i}_L(u_c) & \text{in } \Omega \setminus U_i, \\ \\ \mathbb{P}^{\Omega\cap U_i}_L(u_c) & \text{in } \Omega \cap U_i. \end{cases} \tag{5.46}$$

Moreover

$$(i) \qquad \left\| \mathbb{P}_L^{U_i \cap \Omega}(u_c) \right\|_{L^q(U_i \cap \Omega)} \leq C \left\| \mu \right\|_{W^{-2/q,q}(\partial\Omega)} ,$$
$$(ii) \qquad \left\| \mathbb{P}_L^{\Omega \backslash U_i}(u_c) \right\|_{L^q(\Omega \backslash U_i)} \leq C \left\| \mu \right\|_{W^{-2/q,q}(\partial\Omega)} .$$

Combining these inequalities with (5.43), (5.44) yields to

$$\left\| \mathbb{P}_L^\Omega(\mu) \right\|_{L^q(\Omega; \rho_{\partial\Omega} dx)} = \left\| u \right\|_{L^q(\Omega; \rho_{\partial\Omega} dx)} \leq C \left\| \mu \right\|_{W^{-2/q,q}(\partial\Omega)} , \tag{5.47}$$

and we finish the proof with the help of a partition of unity. The proof of (5.17) is the same as in Step 2. □

Remark. By using sharp estimates on the Green kernel of a general elliptic operator in a general smooth domain it can be checked directely that (5.17) is valid for any signed boundary q admissible measure. However, it is not known if the implication

$$\mu \in \mathfrak{M}(\partial\Omega) \cap W^{-2/q,q}(\partial\Omega) \Longrightarrow \mu \text{ is boundary } q\text{-admissible}, \tag{5.48}$$

holds.

It is proven in [75] that Proposition 5.9 admits an extension in the framework of Besov spaces $B^{-s,q}$ (see e.g. [93]). When s is not an integer or $q = 2$, the Besov space $B^{-s,q}$ coincides with the Sobolev space $W^{-s,q}$.

Proposition 5.10 *Let $s > 0$, $q > 1$ and μ be a distribution on S^{n-1}. Then*

$$\mu \in B^{-s,q}(S^{n-1}) \Longleftrightarrow \mathbb{P}_{-\Delta}^B(\mu) \in L^q(B; (1 - |x|)^{sq-1} dx).$$

Moreover there exists a constant $C > 0$ such that for any $\mu \in B^{-s,q}(S^{n-1})$,

$$C^{-1} \|\mu\|_{B^{-s,q}(S^{n-1})} \leq \left(\int_B \left| \mathbb{P}_{-\Delta}^B(\mu) \right|^q (1 - |x|)^{sq-1} dx \right)^{1/q} \leq C \|\mu\|_{B^{-s,q}(S^{n-1})}. \tag{5.49}$$

The dual form of Proposition 5.9 is the following,

Proposition 5.11 *Let $q \geq (n+1)/(n-1)$ and the assumptions on L and Ω be satisfied as in Proposition 5.9. Then*

$$\varphi \in L^{q'}(\Omega; \rho_{\partial\Omega}^{-q'/q} dx) \Longleftrightarrow \frac{\partial}{\partial \mathbf{n}_{L^*}} \mathbb{G}_{L^*}^\Omega(\varphi) \in W^{2/q,q'}(\partial\Omega).$$

Moreover there exists a constant $C > 0$ such that, for any $\varphi \in L^{q'}(\Omega; \rho_{\partial\Omega}^{-q'/q}) dx)$,

$$C^{-1} \|\varphi\|_{L^{q'}(\Omega; \rho_{\partial\Omega}^{-q'/q} dx)} \leq \left\| \frac{\partial}{\partial \mathbf{n}_{L^*}} \mathbb{G}_{L^*}^\Omega(\varphi) \right\|_{W^{2/q,q'}(\partial\Omega)} \leq C \|\varphi\|_{L^{q'}(\Omega; \rho_{\partial\Omega}^{-q'/q} dx)}. \tag{5.50}$$

Proof. Let $\mu \in \mathfrak{M}(\partial\Omega)$. By duality between $L^q(\Omega;\rho_{\partial\Omega}dx)$ and $L^{q'}(\Omega;\rho_{\partial\Omega}dx)$, we write

$$\int_\Omega \mathbb{P}_L^\Omega(\mu)\psi\rho_{\partial\Omega}dx = \int_\Omega \mathbb{P}_L^\Omega(\mu)L^*\zeta dx = -\int_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}d\mu, \tag{5.51}$$

where $\zeta = \mathbb{G}_{L^*}^\Omega(\psi\rho_{\partial\Omega})$. Then the adjoint operator $\left[\mathbb{P}_L^\Omega\right]^*$ of $\mathbb{P}_L^\Omega$ is defined by

$$\left[\mathbb{P}_L^\Omega\right]^*(\psi) = -\frac{\partial}{\partial\mathbf{n}_{L^*}}\mathbb{G}_{L^*}^\Omega(\psi\rho_{\partial\Omega}). \tag{5.52}$$

Consequently, Proposition 5.9 implies that there exists a constant $C>0$ such that

$$C^{-1}\|\psi\|_{L^{q'}(\Omega;\rho_{\partial\Omega}dx)} \leq \left\|\frac{\partial}{\partial\mathbf{n}_{L^*}}\mathbb{G}_{L^*}^\Omega(\rho_{\partial\Omega}\psi)\right\|_{W^{2/q,q'}(S^{n-1})} \leq C\|\psi\|_{L^{q'}(\Omega;\rho_{\partial\Omega}dx)}. \tag{5.53}$$

But

$$\psi \in L^{q'}(\Omega;\rho_{\partial\Omega}dx) \Longleftrightarrow \rho_{\partial\Omega}\psi \in L^{q'}(\Omega;\rho_{\partial\Omega}^{(1-q')}dx).$$

Putting $\varphi = \rho_{\partial\Omega}\psi$, implies (5.50). □

Proof of Theorem 5.8. (i) Assume that u is a solution of (5.15). Then $u \in L^q(\Omega;\rho_{\partial\Omega}dx)$, and for any $\zeta \in C_c^{1,L}(\overline{\Omega})$, there holds

$$\begin{aligned}\left|\int_{\partial\Omega}\frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}d\mu\right| &= \left|\int_\Omega\left(uL^*\zeta+\zeta|u|^{q-1}u\right)dx\right|,\\ &\leq \|u\|_{L^q(\Omega;\rho_{\partial\Omega}dx)}\|L^*\zeta\|_{L^{q'}(\Omega;\rho_{\partial\Omega}^{-q'/q}dx)} + \int_\Omega |u|^q|\zeta|\,dx,\\ &\leq \|u\|_{L^q(\Omega;\rho_{\partial\Omega}dx)}\left\|\frac{\partial\zeta}{\partial\mathbf{n}_{L^*}}\right\|_{W^{2/q,q'}(\partial\Omega)} + \int_\Omega |u|^q|\zeta|\,dx,\end{aligned} \tag{5.54}$$

since $\mathbb{G}_{L^*}^\Omega(L^*\zeta)=\zeta$. Let $\eta \in W^{2/q,q'}\partial\Omega$, and, for $\delta>0$, put $\zeta = \delta^{-2}\rho_{\partial\Omega}(\delta-\rho_{\partial\Omega})_+^2\mathbb{P}_L^\Omega(\eta)$,

$$\left|\int_{\partial\Omega}\eta d\mu\right| \leq \|u\|_{L^q(\Omega;\rho_{\partial\Omega}dx)}\|\eta\|_{W^{2/q,q'}(\partial\Omega)} + \delta^{-2}\int_\Omega \rho_{\partial\Omega}(\delta-\rho_{\partial\Omega})_+^2\left|\mathbb{P}_L^\Omega(\eta)\right||u|^q\,dx. \tag{5.55}$$

Let $K\subset\partial\Omega$ be a compact subset such that $C_{2/q,q'}(K)=0$. Then there exists a sequence $\{\eta_n\}\subset W^{2/q,q'}(\partial\Omega)$ with the property that $0\leq\eta_n\leq 1$, $\eta_n\equiv 1$ in a neigborhood of K and $\eta_n\to 0$ in $W^{2/q,q'}(\partial\Omega)$ as $n\to\infty$. We take $\eta=\eta_n$ in (5.55). Since $u\in L^q(\Omega;\rho_{\partial\Omega}dx)$ and K has measure zero, the two terms in the right-hand side of (5.55) converge to 0 when $n\to\infty$. Thus μ does not charge Borel subsets with $C^{2/q,q'}$-capacity zero. It follows that μ is the sum of an integrable function and a mesure in $W^{-2/q,q}(\partial\Omega)$, by Corollary 3.18.

(ii) Conversely, let μ be a boundary measure which does not charge Borel subsets with $C_{2/q,q'}$-capacity zero. Assuming first that μ is positive, by Proposition 3.17 there exists an increasing sequence $\{\mu_j\}$ of elements of $W^{-2/q,q}(\partial\Omega)\cap\mathfrak{M}_+(\partial\Omega)$ which converges to μ. By Proposition 5.9, the μ_j are boundary-q-admissible and the sequence $\{u_j\}$ of solutions of

$$\begin{aligned} Lu_j+|u_j|^{q-1}u_j &= 0 \quad \text{in } \Omega,\\ u_j &= \mu_j \quad \text{on } \partial\Omega,\end{aligned} \tag{5.56}$$

is increasing. Moreover $u_j \geq 0$. If $u = \lim_{j\to\infty} u_j$, then $u \geq 0$. Since

$$\int_\Omega \left(u_j L^* \zeta + u_j^q \zeta \right) dx = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}_{L^*}} d\mu_j, \tag{5.57}$$

for any $\zeta \in C_c^{1,L}(\overline{\Omega})$. Taking $\zeta = \eta_1$, the solution of

$$\begin{aligned} L^*\eta_1 &= 1 \quad \text{in } \Omega, \\ \eta_1 &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

we deduce $u \in L^1(\Omega) \cap L^q(\Omega; \rho_{\partial\Omega} dx)$ by the monotone convergence Theorem. Therefore (5.57) implies that u is the solution of

$$\begin{aligned} Lu + |u|^{q-1} u = 0 \quad &\text{in } \ \Omega, \\ u = \mu \quad &\text{on } \ \partial\Omega. \end{aligned} \tag{5.58}$$

If μ is a signed measure, we procede as in the proof of Theorem 3.20, by truncating the nonlinearity and inroducing the solutions of (5.15) associated to μ_+ and $-\mu_-$ on the boundary. □

5.2 Boundary singularities

5.2.1 Isolated singularities

The study of boundary singularities of solutions of semilinear elliptic equations started with the work of Gmira and Véron [48]. As in the case of equations with internal singularities, the starting idea is to study the model case where $\Omega = \mathbb{R}^n_+$, $\partial\Omega = \partial\mathbb{R}^n_+ \approx \mathbb{R}^{n-1}$ and the singularity is located at $x = 0$. In spherical coordinates $x = (r, \sigma)$ where $r > 0$, $\sigma \in S^{n-1}$, the existence of a solution u to

$$-\Delta u + |u|^{q-1} u = 0, \tag{5.59}$$

in $\mathbb{R}^n_+$ $(q > 1)$ which vanishes on $\partial\mathbb{R}^n_+ \setminus \{0\}$ is enlighted if we look for it under the separable form $u(r, \sigma) = r^\alpha \omega(\sigma)$. Then $\alpha = -2/(q-1)$ and ω is a solution of

$$-\Delta_{S^{n-1}} \omega - \left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1} - n\right)\omega + |\omega|^{q-1}\omega = 0 \quad \text{on } \ S^{n-1}_+ = S^{n-1} \cap \mathbb{R}^n_+, \tag{5.60}$$

which vanishes on the equator $\partial S^{n-1}_+ \approx S^{n-2}$. Since the first nonzero eigenvalue of the Laplace-Beltrami operator in $W^{1,2}_0(S^{n-1}_+)$ is $n-1$, it is clear, by multiplying (5.60) by ω and integrating over S^{n-1}_+, that no nontrivial solution of (5.60) exists whenever $(2/(q-1))(2q/(q-1) - n) \leq n - 1$. Equivalently $q \geq (n+1)/(n-1)$. Conversely, if $(2/(q-1))(2q/(q-1) - n) < n - 1$ solutions to (5.60) exist. The stable solutions are obtained by minimizing the functional

$$\eta \mapsto J(\eta) = \int_{S^{n-1}_+} \left(\frac{1}{2} \left|\nabla_{S^{n-1}} \eta\right|^2 - \left(\frac{1}{q-1}\right)\left(\frac{2q}{q-1} - n\right)\eta^2 + \frac{1}{q+1}|\eta|^{q+1} \right) d\sigma, \tag{5.61}$$

over the space $W_0^{1,2}(S_+^{n-1})$, where $\nabla_{S^{n-1}}$ denotes the covariant derivative identified with the tangential gradient thanks to the isometrical imbedding of S^{n-1} into $\mathbb{R}^n$. Put

$$\Lambda_{q,n} = \left(\left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1}-n\right)\right) = \ell_{q,n}^{q-1},$$

and let $\mathcal{S}_+$ be the set of solutions of (5.60) in S_+^{n-1} which vanishes on ∂S^{n-1}. As we have already seen it, if $q \geq (n+1)/(n-1)$ this set is reduced to $\{0\}$. Conversely, if $1 < q < (n+1)/(n-1) \iff \Lambda_{q,n} > n-1$, there exist minimizing solutions to (5.60). Besides this fact, the positive solutions are unique. Moreover, if $\Lambda_{q,n} \leq 2n$, which is the second eigenvalue of $\Delta_{S^{n-1}}$ in $W_0^{1,2}(S_+^{n-1})$, all the solutions of (5.60) vanishing on the equator have constant sign. Finally, if $\Lambda_{q,n} > 2n$ there exist changing sign solutions.

Let Ω be an open subset of $\mathbb{R}^n$ with a boundary of class $C^{2,\theta}$ for some $\theta \in (0,1)$, and $0 \in \partial\Omega$. It can be performed an orthogonal change of coordinates in $\mathbb{R}^n$ in order the axis $\{x : x_j = 0,\ \forall j = 1,...,n-1\}$ be the normal direction to $\partial\Omega$, $\mathbf{e}_n$ be the unit outward normal vector at 0 and $\partial\mathbb{R}_+^n \approx \mathbb{R}^{n-1}$ the tangent plane to $\partial\Omega$ at 0. Let u be any solution to (5.59) in Ω which is continuous in $\overline{\Omega}\setminus\{0\}$ and coincides on $\partial\Omega\setminus\{0\}$ with a function $g \in C(\partial\Omega)$. For $R > 0$ small enough and $m_+ = \max\{g(x) : x \in \partial\Omega \cap B_R\}$, the function

$$x \mapsto \tilde{u}(x) = \begin{cases} (u(x) - m_+)_+ & \text{if } x \in \Omega \cap B_R, \\ 0 & \text{if } x \in B_R \setminus \overline{\Omega}, \end{cases}$$

is a subsolution of (5.59) in $B_R \setminus \{0\}$. But the Keller-Osserman estimate implies

$$u(x) \leq m_+ + C\,|x|^{-2/q-1}, \forall x \in \overline{\Omega} \cap B_R \setminus \{0\},$$

for some $C = C(n,q,R) > 0$. In the same way, u is bounded from below in the same set by $m_- - C\,|x|^{-2/q-1}$, where $m_- = \min\{g(x) : x \in \partial\Omega \cap B_R\}$. Hence the function $x \mapsto |x|^{2/q-1}\,u(x)$ is uniformly bounded in $\overline{\Omega}\cap B_R\setminus\{0\}$. We perform a change of coordinates $y = \phi(x)$ which transforms $\Omega \cap B_R$ into $\mathbb{R}_+^n \cap B_R$ and $\partial\Omega \cap B_R$ into $R^{n-1} \cap B_R$. We define v by

$$y \mapsto v(y) = v(r,\sigma) = \left|\phi^{-1}(y)\right|^{2/(q-1)} u(\phi^{-1}(y)), \quad (r,\sigma) \in (0,R) \times S_+^{n-1},$$

and put $w(t,\sigma) = v(r,\sigma)$ with $t = \ln r$. Then w satisfies an equation of the type

$$\begin{aligned} 0 = (1+\epsilon_1(t))w_{tt} &+ \left(n - 2\frac{q+1}{q-1} + \epsilon_2(t)\right) w_t + (\Lambda_{q,n} + \epsilon_3(t))\,w + \Delta_{S^{n-1}} w \\ &+ \langle \nabla_{S^{n-1}} w.\epsilon_4(t)\rangle + + \langle \nabla_{S^{n-1}} w_t.\epsilon_5(t)\rangle + \langle \nabla_{S^{n-1}} \langle \nabla_{S^{n-1}} w.\mathbf{e}_n\rangle.\epsilon_6(t)\rangle + |w|^{q-1}\,w \end{aligned} \tag{5.62}$$

in $(-\infty, \ln R] \times S_+^{n-1}$, where the $\epsilon_j(t)$ depend on the change of coordinates and verify

$$|\epsilon_j(t,\sigma)| \leq C_j e^t, \quad \forall (t,\sigma) \in (-\infty, \ln R] \times S_+^{n-1}, \quad j = 1,...,6. \tag{5.63}$$

Since $|w(t,\sigma)| \leq Ce^{2qt/(q-1)}$, we can use the elliptic equations regularity theory and a Lyapounov style analysis at $-\infty$. The following result is due to Gmira and Véron [48].

Theorem 5.12 *Suppose* $1 < q < (n+1)/(n-1)$. *Then, with the previous notations, there exists a compact connected subset* $\mathcal{E}_+$ *of the set of the solutions of (5.60) in* S_+^{n-1} *which vanish on* ∂S_+^{n-1}, *such that*

$$\lim_{t\to-\infty} \mathrm{dist}_{\mathrm{C^2(S_+^{n-1})}}(\mathrm{w(t,.)}, \mathcal{E}_+) = 0, \tag{5.64}$$

where $dist_{C^2(S_+^{n-1})}$ *denotes the distance associated with the* $C^2(S_+^{n-1})$*-norm. Moreover, the set* $\mathcal{E}_+$ *is reduced to a singleton in the following cases :*
(i) u *is nonnegative,*
(ii) $(n+2)/2n \leq q < (n+1)/(n-1)$,
(iii) $n = 2$.

When $\mathcal{E}_+ = \{0\}$ it is possible to make more precise the way the function $w(t,.)$ converges to 0 as $t \to -\infty$. By adapting the method developed in [27], it is proven in [48] that the following result holds,

Theorem 5.13 *Suppose* $1 < q < (n+1)/(n-1)$ *and let* w *be the solution of (5.62) associated to* u, *solution of (5.59). Assume*

$$\lim_{t\to-\infty} \|w(t,.)\|_{C^2(S_+^{n-1})} = 0.$$

Then, if one of the following conditions holds :
(a) u *is nonnegative,*
(b) $n = 2$ *and* $\partial\Omega$ *is locally a straight line near* 0,
(c) $2/(q-1)$ *is not an integer,*
(i) either u *can be extended to* $\overline{\Omega}$ *as continuous function solution of the Dirichlet problem*

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{5.65}$$

(ii) or there exists an integer $k \in [n-1, 2/(q-1))$ *and a nonzero solution* ψ *of*

$$\begin{aligned} \Delta_{S^{n-1}} \psi + k(n+k-2)\psi &= 0 \quad \text{in } S_+^{n-1}, \\ \psi &= 0 \quad \text{on } \partial S_+^{n-1}, \end{aligned} \tag{5.66}$$

such that

$$\lim_{t\to-\infty} e^{(k-2/(q-1))t} w(t,.) = \psi, \tag{5.67}$$

in the $C^2(S_+^{n-1})$*-topology.*

The meaning of this result is the following : either u has a strong boundary singularity which is described thanks to the set $\mathcal{S}_+$ of solutions of (5.60) vanishing on the equator, either there exists a spherical harmonic of degree k such that

$$\lim_{\substack{x \to 0 \\ x/|x| \to \sigma}} |x|^k u(x) = \psi(\sigma), \quad \text{uniformly for } \sigma \in S_+^{n-1}, \tag{5.68}$$

or u is regular function.

When $-\Delta$ is replaced by an elliptic operator L with variable Lipschitz continuous coefficients, most of the above results extend in the same way as for the isolated internal singularities (see the section on isolated singularities).

5.2.2 Removable singularities

The first result on removability (see [48]) is the following.

Theorem 5.14 *Let Ω be a C^2 domain in $\mathbb{R}^n$, x_0 a boundary point, and g a continuous real valued function defined on $\Omega \times \mathbb{R}$, such that*

$$\liminf_{r\to\infty} \frac{g(x,r)}{r^{(n+1)/(n-1)}} > 0 \quad \textit{and} \quad \limsup_{r\to\infty} \frac{g(x,r)}{|r|^{(n+1)/(n-1)}} < 0, \quad \forall x \in \Omega, \tag{5.69}$$

uniformly with respect to $x \in \Omega$. If $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{x_0\})$ is a solution of

$$-\Delta u + g(x,u) = 0 \quad \textit{in } \Omega, \tag{5.70}$$

which coincides on $\partial\Omega \setminus \{x_0\}$ with some $\phi \in C(\partial\Omega)$, then u can be extended as a $C(\overline{\Omega})$ function, which verifies

$$\begin{aligned} -\Delta u + g(x,u) &= 0 \quad \textit{in } \Omega, \\ u &= \phi \quad \textit{on } \partial\Omega. \end{aligned} \tag{5.71}$$

Actually, their proof could have been adapted, without any deep modification, to Equation (5.1). A much more general result will be given later on.

Definition 5.15 Let Ω be a C^2 domain in $\mathbb{R}^n$ and $q \geq (n+1)/(n-1)$.
(i) A Borel subset K of $\partial\Omega$ is said q-removable if any nonnegative function $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus K)$ solution of (5.59) which vanishes on $\partial\Omega$ is identically zero.
(ii) A Borel subset K of $\partial\Omega$ is said conditionally q-removable if any nonnegative function $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus K)$ solution of (5.59) belongs to $L^q_{loc}(\overline{\Omega}; \rho_{\partial\Omega} dx)$.

The condition $q \geq (n+1)/(n-1)$ is necessary, since, below this value, only the empty set is removable by Theorem 5.6. The main removability result is the folllowing,

Theorem 5.16 *Let Ω be a C^2 bounded domain in $\mathbb{R}^n$, $q \geq (n+1)/(n-1)$ and $K \subset \Omega$ be compact. Then the following assertions are equivalent.*
(i) K is q-removable.
(ii) K is conditionally q-removable.
(iii) $C_{2/q,q'}(K) = 0$.

This result was first proven by Le Gall [63] in the case $q = 2$, by probabilistic methods, then by Dynkin and Kuznetsov [36] in the case $q \leq 2$, by a combination of analytic and probabilistic methods and by Marcus and Véron [71] when $q > 2$ with purely analytic tools. All the proof are based upon the construction of suitable lifting operators which transform functions defined on the boundary into functions defined in Ω. In [72] the first unified proof, valid in all the cases $q \geq (n+1)/(n-1)$ is given. We shall present a sketch of it below.

Definition 5.17 A linear map $R : C^2(\Omega) \mapsto C^2(\overline{\Omega})$ is called a positive lifting if

$$R(\eta)\big|_{\partial\Omega} = \eta \quad \text{and} \quad \eta \geq 0 \Longrightarrow R(\eta) \geq 0. \tag{5.72}$$

Lemma 5.18 *Let ϕ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(\Omega)$ and $q \geq (n+1)/(n-1)$. There exists a positive lifting operator $R : \eta \mapsto R(\eta) = R_\eta$ with the additional property*

$$\|R_\eta\|_{L^\infty(\Omega)} \leq \|\eta\|_{L^\infty(\partial\Omega)},$$

and

$$\left\| \left|\phi^{1/q'}\Delta R_\eta\right| + 2\left|\phi^{-1/q}\langle\nabla R_\eta.\nabla\phi\rangle\right| \right\|_{L^{q'}(\Omega)} \leq C\|\eta\|_{W^{2/q,q'}(\partial\Omega)}, \quad \forall \eta \in W^{2/q,q'}(\partial\Omega). \tag{5.73}$$

Furthermore

$$\begin{aligned} \left\| \left|\phi^{1/q'}R_\eta\Delta R_\eta\right| + 2\left|\phi^{-1/q}R_\eta\langle\nabla R_\eta.\nabla\phi\rangle\right| + \phi^{1/q'}\left|\nabla R_\eta\right|^2 \right\|_{L^{q'}(\Omega)} \\ \leq C(1 + \|\eta\|_{W^{2/q,q'}(\partial\Omega)}), \quad \forall \eta \in \mathcal{T}^*, \end{aligned} \tag{5.74}$$

where $\mathcal{T}^* = \{\eta \in W^{2/q,q'}(\partial\Omega) : 0 \leq \eta \leq 1\}$.

Proof. In Section 2.4 we have already introduced the foliation of $\partial\Omega$ by the Σ_β

$$\Sigma_\beta := \{x \in \Omega : \rho_\Omega(x) = \beta\}, \quad 0 < \beta \leq \beta_0,$$

for β_0 depending on the curvature of $\partial\Omega$, with $\Sigma_0 = \Sigma = \partial\Omega$, $\Omega_\beta = \{x \in \Omega : \rho_{\partial\Omega}(x) > \beta\}$ and $G_\beta = \Omega \setminus \overline{\Omega}_\beta$. For every $0 < \beta \leq \beta_0$ and $x \in G_\beta$ there exists a unique $\sigma(x) \in \Sigma$ such that $|x - \sigma(x)| = \rho_{\partial\Omega}(x)$, and the correspondence $x \longleftrightarrow (\rho_{\partial\Omega}(x), \sigma(x))$ defines a smooth change of coordinates near the boundary called the flow coordinates. In terms of flow ccordinates, the Laplacian has the following form

$$\Delta = \frac{\partial^2}{\partial\rho^2} + b_0\frac{\partial}{\partial\rho} + \Lambda_\Sigma,$$

where ρ stands for $\rho_{\partial\Omega}$, b_0 depends on x and Λ_Σ is a second order elliptic operator on Σ with coefficients depending also on x. Moreover

$$\Lambda_\Sigma \to \Delta_\Sigma \quad \text{and} \quad b_0 \to \kappa \quad \text{as} \quad \rho_{\partial\Omega}(x) \to 0,$$

where Δ_Σ is the Laplace-Beltrami operator on Σ, and κ the mean curvature of Σ (see [13]). If $\eta \in C(\Sigma)$, let $H = H_\eta$ be the solution of the initial value problem

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \Delta_\Sigma H \quad \text{in } \mathbb{R}_+ \times \Sigma, \\ H(0,\cdot) &= \eta(\cdot) \quad \text{in } \Sigma. \end{aligned} \tag{5.75}$$

We can express H in terms of the two coordinates (τ, σ). Let $h \in C^\infty(\mathbb{R}_+)$ be a truncation function with value in $[0,1]$, $h \equiv 1$, on $[0, \beta_0/2]$ and $h \equiv 0$, on $[\beta_0, \infty)$. The lifting $R = R_\eta$ of η is defined by

$$R_\eta(x) = \begin{cases} H_\eta(\phi^2(x), \sigma(x))h(\rho_{\partial\Omega}(x)), & \forall x \in G_{\beta_0} \\ 0, & \forall x \in \Omega_{\beta_0}. \end{cases} \tag{5.76}$$

Clearly the positivity and contraction principle in uniform norms hold. The proof of (5.73) and (5.74) is much more elaborated and settled upon analytic semigroups theory and delicate interpolation results (see [72] for a detailled proof). $\square$

Proof of Theorem 5.16. (iii)$\Longrightarrow$ (ii) Let

$$\mathcal{T}_K = \{\eta \in C^2(\partial\Omega) : 0 \le \eta \le 1,\ \eta \equiv 0 \text{ in an open relative neighborhood of } K\}.$$

Put $\zeta_\eta := \phi R_\eta^{2q'}$. Then $0 \le \zeta \le \phi$, and $\zeta_\eta(x) = O\left((\rho_{\partial\Omega}(x))^{1+2q'}\right)$ in a neighborhood V_η of K. Since in the case of Equation (5.59), the Keller-Osserman *a priori* bound implies

$$|u(x)| \le C(N,q)(\rho_{\partial\Omega}(x))^{-2/(q-1)}, \quad \forall x \in \Omega, \tag{5.77}$$

and $u(x) = O\left(\rho_{\partial\Omega}(x)\right)$ if $\rho_{\partial\Omega}(x) \to 0$, outside V_η, we derive

$$u^q(x)\zeta_\eta(x) = O\left(\rho_{\partial\Omega}(x)\right) \quad \text{in } \Omega. \tag{5.78}$$

Moreover, if λ_1 is the eigenvalue corresponding to ϕ,

$$\begin{aligned} \Delta\zeta_\eta &= -\lambda_1\phi R_\eta^{2q'} + \phi\Delta R_\eta^{2q'} + 2\langle\nabla\phi.\nabla R_\eta^{2q'}\rangle \\ &= -\lambda_1\zeta_\eta + 2q'\phi R_\eta^{2q'-1}\Delta R_\eta + 2q'(2q'-1)R_\eta^{2q'-2}\left|\nabla R_\eta\right|^2 + 2q' R_\eta^{2q'-1}\langle\nabla\phi.\nabla R_\eta\rangle. \end{aligned} \tag{5.79}$$

Therefore

$$u\left|\Delta\zeta_\eta\right| \le C(\eta)uR_\eta^{2q'-2}.$$

Because $\eta \in \mathcal{T}_K$, $u\Delta\zeta_\eta$ remains bounded in Ω. For $0 < \beta \le \beta_0$,

$$\int_{\Omega\setminus G_\beta} \zeta_\eta\Delta u\, dx = \int_{\Omega\setminus G_\beta} u\Delta\zeta_\eta\, dx + \int_{\Sigma_\beta}\left(\zeta_\eta\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial\zeta_\eta}{\partial\mathbf{n}}\right) dS, \tag{5.80}$$

and combining (5.77) with Schauder estimates,

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Sigma_\beta} = O(\beta^{-(q+1)/(q-1)}),$$

hence

$$\Big(\zeta_\eta \frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial \zeta_\eta}{\partial \mathbf{n}}\Big)\Big|_{\Sigma_\beta} = O(\beta).$$

Letting $\beta \to 0$ in (5.80) implies

$$\int_\Omega \Big(- u\Delta\zeta_\eta + u^q\zeta_\eta\Big)dx = 0. \tag{5.81}$$

By Hölder's inequality,

$$\begin{aligned}\int_\Omega u\,|\Delta\zeta_\eta|\,dx &\le \Big(\int_\Omega u^q\zeta_\eta\,dx\Big)^{1/q}\Big(\int_\Omega \zeta_\eta^{-q'/q}\,|\Delta\zeta_\eta|^{q'}\,dx\Big)^{1/q'}\\ &\le c\Big(\int_\Omega u^q\zeta_\eta\,dx\Big)^{1/q}\Big(\int_\Omega (\zeta_\eta + M(\eta)^{q'})\,dx\Big)^{1/q'},\end{aligned} \tag{5.82}$$

where

$$M(\eta) = \Big|\phi^{1/q'}R_\eta\Delta R_\eta\Big| + 2\Big|\phi^{-1/q}R_\eta\langle\nabla R_\eta.\nabla\phi\rangle\Big| + \phi^{1/q'}\,|\nabla R_\eta|^2\,.$$

Since by Lemma 5.18,

$$\|M(\eta)\|_{L^{q'}(\Omega)} \le C_1(1 + \|\eta\|_{W^{2/q,q'}(\partial\Omega)}),$$

it follows from (5.81) and (5.82),

$$\int_\Omega u^q\zeta_\eta dx \le C_2(1 + \|\eta\|_{W^{2/q,q'}(\partial\Omega)})^{q'}. \tag{5.83}$$

If we put $\eta^* = 1-\eta$, then $\|\eta\|^{q'}_{W^{2/q,q'}(\partial\Omega)} \le C' + \|\eta^*\|^{q'}_{W^{2/q,q'}(\partial\Omega)}$. If K has $C_{2/q,q'}$-capacity zero, there exists a sequence $\{\eta_n^*\} \subset C^2(\partial\Omega)$ such that $0 \le \eta_n^* \le 1$, $\eta_n^* \equiv 1$ in a relatively open neighborhood of K and

$$\|\eta_n^*\|_{W^{2/q,q'}(\partial\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Since a boundary set with $C_{2/q,q'}$-capacity zero has zero $(n-1)$ -Hausdorff measure, $\eta_n^* \to 0$ as $n \to \infty$. Thus $\zeta_{\eta_n^*} \to \phi$. If we let $n \to \infty$ in (5.83) we finally obtain

$$\int_\Omega u^q\phi dx \le C_2, \tag{5.84}$$

with $C_2 = C_2(K)$. Thus K is conditionally q-removable.

(ii) $\Longrightarrow$ (i) Since $u^q \in L^1(\Omega;\rho_{\partial\Omega}dx)$, $u \ge 0$ and

$$-\Delta u = -u^q,$$

the function $v = u + \mathbb{G}^\Omega_{-\Delta}(u^q)$ is positive and harmonic in Ω, thus it admits a boundary trace $\mu \in \mathfrak{M}_+(\partial\Omega)$. Since the boundary trace of $\mathbb{G}^\Omega_{-\Delta}(u^q)$ is the zero measure, it is infered that u admits the same boundary trace μ, the support of which is included into the set K. Moreover

$$0 \leq u = \mathbb{P}^{\Omega}_{-\Delta}(\mu) - \mathbb{G}^{\Omega}_{-\Delta}(u^q) \leq \mathbb{P}^{\Omega}_{-\Delta}(\mu).$$

Therefore $u = u_\mu$, solution of Problem (5.15) with $L = -\Delta$. Consequently μ does not charge boundary sets with $C_{2/q,q'}$-capacity zero and the same property is shared by $k\mu$, for any $k \in \mathbb{N}_*$. Put $u_k = u_{k\mu}$. If μ is not zero, the sequence of solutions $\{u_k\}$ is increasing and converges to some u_∞ when $k \to \infty$. Because u_k vanishes on $\partial\Omega \setminus K$, it follows from the Keller-Osserman construction that u_∞ inherits the same property. Furthermore

$$\int_\Omega \left(-u_k \Delta\zeta_{\eta^*} + \zeta_{\eta^*} u_k^q\right) dx = -k \int_{\partial\Omega} \frac{\partial \zeta_{\eta^*}}{\partial \mathbf{n}} d\mu, \tag{5.85}$$

where $\eta \in \mathcal{T}$, $\eta^* = 1 - \eta$ and $\zeta_{\eta^*} = \phi R^{2q'}_{\eta^*}$. Because μ is not zero, the right-hand side of (5.85) tends to infinity with k. Since K is conditionally q-removable $u_\infty \in L^1(\Omega; \rho_{\partial\Omega} dx)$. Moreover, as we have seen it before,

$$\left|\int_\Omega u_k \Delta\zeta_{\eta^*} dx\right| \leq C \left(\int_\Omega u_k^q \phi dx\right)^{1/q} \left(1 + \|\eta^*\|_{W^{2/q,q'}(\partial\Omega)}\right).$$

Hence, the right-hand side of (5.85) is bounded independently of k, which is a contradiction.

(i) $\Longrightarrow$ (iii). If we assume $C_{2/q,q'}(K) > 0$, there exists a measure $\mu_K \in \mathfrak{M}_+(\partial\Omega) \cap W^{-2/q,q}(\partial\Omega)$, satisfying $\mu_K(\partial\Omega \setminus K) = 0$ and $C_{2/q,q'}(K) = \mu_K(K)$. This measure is an extremal for the dual definition of the capacity of K (already introduced in (3.54 with Bessel potentials) :

$$C_{2/q,q'}(K) = \sup_{\substack{\mu \in \mathfrak{M}_+(\partial\Omega) \\ \mu(\partial\Omega \setminus K) = 0}} \left(\frac{\mu(K)}{\left\|\mathbb{P}^{\Omega}_{-\Delta}(\mu)\right\|_{L^q(\Omega;\rho_{\partial\Omega} dx)}}\right)^{q'},$$

see [1, Th. 2.2.7]. Hence Problem (5.15) with $L = -\Delta$ is solvable with $\mu = \mu_K$, thus K is not conditionally q-removable. □

5.3 The boundary trace problem

One of the most striking aspects in the study on positive solutions of (5.15) in a domain Ω relies on the possibility of defining a boundary trace which is no longer a Radon measure, but a generalized Borel measure, that is a measure which can take infinite values on compact boundary subsets. The second important task of the theory of boundary trace is to analyse the connection between the set of all the boundary traces and the set of solutions. These notions were first studied by Le Gall [61], [62] in the case $L = -\Delta$, $q = n = 2$, and then extended by Marcus and Véron [68], [69], [70]. For simplicity we shall consider first the model case

$$-\Delta u + |u|^{q-1} u = 0 \quad \text{in } \Omega. \tag{5.86}$$

We adopt the notations of Section 2.4

Theorem 5.19 *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and $q > 1$. Let u be a positive solution of (5.86). Then for any $a \in \partial\Omega$ the following dichotomy holds :*
(i) either for every relatively open subset $\mathcal{O} \subset \Omega$ containing a,

$$\lim_{t\to 0} \int_{\mathcal{O}_t} u(x) dS_t = \infty, \tag{5.87}$$

(ii) or there exist a relatively open subset $\mathcal{O} \subset \Omega$ containing a and a positive linear functional ℓ on $C_c^\infty(\mathcal{O})$ such that for every $\theta \in C_c^\infty(\mathcal{O})$,

$$\lim_{t\to 0} \int_{\mathcal{O}_t} u(x)\theta(x) dS = \ell(\theta). \tag{5.88}$$

Proof. The proof of this result is settled upon the following alternative which holds for every boundary point a :
(I) either there exists an open ball $B_{r_0}(a)$ such that

$$\int_{B_{r_0}(a)\cap\Omega} u^q \rho_{\partial\Omega} dx < \infty, \tag{5.89}$$

(II) or for any $r > 0$,

$$\int_{B_r(a)\cap\Omega} u^q \rho_{\partial\Omega} dx = \infty. \tag{5.90}$$

If (I) holds, let $\epsilon > 0$ and $\mathcal{U}_\epsilon$ be a smooth open subdomain of $\Omega \cap B_{r_0}(a)$ containing $\overline{B}_{r-\epsilon}(a)\Omega$ and such that

$$\overline{B}_{r-\epsilon}(a) \cap \partial\Omega \subset \overline{\mathcal{U}}_\epsilon \cap \partial\Omega \subset B_r(a) \cap \partial\Omega.$$

The function $\tilde{u} = u|_{\mathcal{U}_\epsilon}$ is a nonnegative solution of (5.86) in $\mathcal{U}_\epsilon$ with $\tilde{u}^q \in L^1(\mathcal{U}_\epsilon; \rho_{\partial\mathcal{U}_\epsilon} dx)$. Thus it admits a boundary trace on $\partial\mathcal{U}_\epsilon$ which belongs to $\mathfrak{M}_+(\partial\mathcal{U}_\epsilon)$. Therefore, for any $\theta \in C_c^\infty(\partial\mathcal{U}_\epsilon)$, there holds

$$\lim_{t\to 0} \int_{\partial\mathcal{U}_{\epsilon\, t}} u(x)\theta(x) dS = \ell_\epsilon(\theta). \tag{5.91}$$

Since ϵ is arbitrary and ℓ_ϵ is uniquely determined on $\partial\mathcal{U}_\epsilon$, assertion (ii) follows.

If (II) holds, let $\eta \in C_c^\infty(\partial\Omega \cap B_r(a))$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $\partial\Omega \cap B_{r/2}(a)$. For $t \in (0, \beta_0/2)$ small enough, we define $\zeta_{\eta,t}$ in the set $\Omega_t \setminus \Omega_{\beta_0}$ by

$$\zeta_{\eta,t}(x) = \zeta_{\eta,t}(\rho_{\partial\Omega}(x) - t, \sigma(x)) = (\phi R_\eta^{2q'})(\rho_{\partial\Omega}(x) - t, \sigma(x)).$$

Then

$$\int_{\Omega_t\setminus\Omega_{\beta_0}} (-u\Delta\zeta_{\eta,t} + u^q\zeta_{\eta,t})\, dx = \int_{\Sigma_t} \eta^{2q'} u dS - \int_{\Sigma_{\beta_0}} \frac{\partial\zeta_{\eta,t}}{\partial\mathbf{n}}(\beta_0 - t, \sigma) dS. \tag{5.92}$$

As we have already seen it

$$\int_{\Omega_t\setminus\Omega_{\beta_0}} |u\Delta\zeta_{\eta,t}|\, dx \le C\|\eta\|_{W^{2/q,q'}} \left(\int_{\Omega_t\setminus\Omega_{\beta_0}} u^q \zeta_{\eta,t} dx\right)^{1/q}.$$

Because the surface integral term in (5.92) on Σ_{β_0} is bounded independently of t, it follows

$$\int_{\Sigma_t} \eta^{2q'} u dS \geq \int_{\Omega_t \setminus \Omega_{\beta_0}} u^q \zeta_{\eta,t} dx - C_1 \|\eta\|_{W^{2/q,q'}} \left(\int_{\Omega_t \setminus \Omega_{\beta_0}} u^q \zeta_{\eta,t} dx \right)^{1/q} - C_2. \tag{5.93}$$

Moreover, as $\eta \equiv 1$ on $\partial\Omega \cap B_{r/2}(a)$, there exists $\delta > 0$ such that $\phi R_\eta^{2q'} \geq \delta$ on $\Omega \cap B_{r/2}(a)$. Hence, by (5.90) and the Beppo-Levi Theorem,

$$\lim_{t \to 0} \int_{\Omega_t \setminus \Omega_{\beta_0}} u^q \zeta_{\eta,t} dx = \infty,$$

which implies

$$\lim_{t \to 0} \int_{\Sigma_t} \eta^{2q'} u dS = \infty, \tag{5.94}$$

and assertion (i) follows. □

We write $\partial\Omega = \mathcal{S}(u) \cup \mathcal{R}(u)$ where $\mathcal{S}(u)$ is the closed subset of boundary points where (i) occurs, and $\mathcal{R}(u) = \partial\Omega \setminus \mathcal{S}(u)$. By using a partition of unity, there exists a unique positive Radon measure μ on $\mathcal{R}(u)$ such that

$$\lim_{t \downarrow 0} \int_{\mathcal{R}(u)} u(\sigma,t) \zeta_t(\sigma,t) dS_t = \int_{\mathcal{R}(u)} \zeta(\sigma) d\mu, \tag{5.95}$$

for every $\zeta \in C_c(\mathcal{R}(u))$. Thus we define the boundary trace by the following identification

$$Tr_{\partial\Omega}(u) = (\mathcal{S}(u), \mu). \tag{5.96}$$

The set $\mathcal{S}(u)$ is called the *singular part* of the boundary trace of u, while $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ is the *regular part*. The couple $(\mathcal{S}(u), \mu)$ defines in a unique way an outer regular positive Borel measure ν (an element of $\mathfrak{B}_+^{reg}(\partial\Omega)$), with singular part $\mathcal{S}(u)$ and regular part μ.

In the subcritical case, an important pointwise characterization of the singular part is the following minoration,

Proposition 5.20 *Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2 boundary $\partial\Omega$, $1 < q < (n+1)/(n-1)$ and u be a positive solution of (5.86) in Ω with boundary trace $(\mathcal{S}(u), \mu)$. If $a \in \mathcal{S}(u)$, then*

$$u(x) \geq u_{\infty a}(x), \quad \forall x \in \Omega, \tag{5.97}$$

where $u_{\infty a} = \lim_{k \to \infty} u_{k\delta_a}$, and $u_{k\delta_a}$ is the solution of

$$\begin{aligned} -\Delta u_{k\delta_a} + |u_{k\delta_a}|^{q-1} u_{k\delta_a} &= 0 \quad \text{in } \Omega, \\ u_{k\delta_a} &= k\delta_a \quad \text{on } \partial\Omega. \end{aligned} \tag{5.98}$$

Proof. Since for any $r > 0$, there holds

$$\lim_{t \to 0} \int_{B_r(a) \cap \Sigma_t} u(x) dS_t = \infty,$$

for any $k > 0$ and $t = t_k = 1/k$, there exists $r_{k,t} > 0$ such that

$$\int_{B_{r_{k,t}}(a)\cap\Sigma_t} u(x)dS_t \geq k.$$

Let m_k be such that

$$\int_{B_{r_{k,t}}(a)\cap\Sigma_t} \min\{m_k, u(x)\}dS_t = k,$$

and denote by v_k the solution of

$$\begin{aligned} -\Delta v_k + |v_k|^{q-1} v_k = 0 \qquad & \text{in } \Omega_t, \\ v_k = \chi_{B_{r_{k,t}}(a)\cap\Sigma_t} \quad & \text{on } \Sigma_t. \end{aligned} \tag{5.99}$$

By the maximum principle, $v_k \leq u$ in Ω_t and by the stability result of Corollary 5.4, v_k converges to $u_{k\delta_a}$ locally uniformly in Ω (actually the proof is given for a fixed domain Ω, but the adaptation to a sequence of expanding smooth domains is straightforward). Thus $u_{k\delta_a} \leq u$ in Ω. Since k is arbitrary, (5.97) follows. □

Remark. Notice that the boundary behaviour of $u_{\infty a}$ is given by Theorem 5.12 : with an apropriate rotation in the space, it is

$$\lim_{\substack{x \to a \\ (x-a)/|x-a| \to \sigma}} |x-a|^{2/(q-1)} u_{\infty a}(x) = \omega(\sigma), \quad \text{uniformly on } S_+^{n-1}, \tag{5.100}$$

where ω is the unique solution of (5.60) on S_+^{n-1} which vanishes on the equator ∂S_+^{n-1}.

The most general boundary value problem concerning positive solutions of (5.86) is to solve the Dirichlet boundary value problem with a given outer regular Borel measure as boundary trace. If $\nu \in \mathfrak{B}_+^{reg}(\partial\Omega)$, we put

$$\mathcal{S} = \mathcal{S}_\nu = \{\sigma \in \partial\Omega : \nu(U) = \infty \text{ for every relatively open neighborhood } U \text{ of } \sigma\}.$$

Clearly $\mathcal{S}_\nu$ is closed and the restriction μ of ν to $\mathcal{R}_\nu = \partial\Omega \setminus \mathcal{S}_\nu$ is a Radon measure. This establishes a one to one correspondence between $\mathfrak{B}_+^{reg}(\partial\Omega)$ and the set of couples $(\mathcal{S}, \mu)$, where $\mathcal{S}$ is a closed subset of $\partial\Omega$ and μ a positive Radon measure on $\mathcal{R} = \partial\Omega \setminus \mathcal{S}$. The following result is proven in [70].

Theorem 5.21 *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and $1 < q < (n+1)/(n-1)$. Then for any $\nu \in \mathfrak{B}_+^{reg}(\partial\Omega)$ with $\nu \approx (\mathcal{S}, \mu)$, where $\mathcal{S}$ is a closed subset of $\partial\Omega$ and μ a positive Radon measure on $\partial\Omega \setminus \mathcal{S}$, there exists a unique solution of*

$$\begin{aligned} -\Delta u + |u|^{q-1} u = 0 \quad & \text{in } \Omega, \\ Tr_{\partial\Omega}(u) = \nu. & \end{aligned} \tag{5.101}$$

Proof. The proof is long and technical, and we shall just indicate the main steps :
(1) By approximation, a minimal solution $\underline{u}_{\mathcal{S},\mu}$ and a maximal solution $\overline{u}_{\mathcal{S},\mu}$ of Problem (5.101) are constructed, so any other solution u satisfies

$$\underline{u}_{\mathcal{S},\mu} \leq u \leq \overline{u}_{\mathcal{S},\mu}. \tag{5.102}$$

(2) Using convexity and the approximations of the minimal and the maximal solutions, it is proven that

$$\overline{u}_{\mathcal{S},\mu} - \underline{u}_{\mathcal{S},\mu} \leq \overline{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}. \tag{5.103}$$

(3) Using (5.77), (5.97), (5.100) and Hopf boundary lemma, there exists $K = K(q,\Omega) > 1$ such that

$$\overline{u}_{\mathcal{S},0} \leq K\underline{u}_{\mathcal{S},0}. \tag{5.104}$$

(4) Assuming that $\underline{u}_{\mathcal{S},0} \neq \overline{u}_{\mathcal{S},\mu}$ (and the strict inequality follows by the strong maximum principle), a convexity argument implies that the function

$$w = \underline{u}_{\mathcal{S},0} - \frac{1}{2K}(\overline{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}),$$

is a supersolution of (5.101) with $\nu \approx (\mathcal{S}, 0)$. Since for $0 < \alpha < 1/(2K)$ $\alpha\underline{u}_{\mathcal{S},0}$ is a subsolution of the same problem with the same boundary trace, and

$$\alpha\underline{u}_{\mathcal{S},0} \leq w,$$

it follows by (Theorem 4.1) that there exists a solution u of (5.86) in Ω and

$$\alpha\underline{u}_{\mathcal{S},0} \leq u \leq w < \underline{u}_{\mathcal{S},0}. \tag{5.105}$$

Because both $\alpha\underline{u}_{\mathcal{S},0}$ and w have the same boundary trace $(\mathcal{S}, 0)$ in the sense of Theorem 5.19, u is a solution of Problem (5.101) with $\nu \approx (\mathcal{S}, 0)$. This fact contradicts the minimality of $\underline{u}_{\mathcal{S},0}$, thus $\overline{u}_{\mathcal{S},0} = \underline{u}_{\mathcal{S},0}$, which, in turn, implies $\overline{u}_{\mathcal{S},\mu} = \underline{u}_{\mathcal{S},\mu}$. □

When $q \geq (n+1)/(n-1)$ neither any positive Radon measure on $\partial\Omega$ is eligible for being the regular part of the boundary trace of a positive solution of (5.86), nor any closed boundary subset for being the singular part : these facts follow from Theorem 5.8 and Theorem 5.16.

Definition 5.22 (i) Let $\mathcal{A}$ be a relatively open subset of $\partial\Omega$ and $\mu \in \mathfrak{M}_+(\mathcal{A})$. Then *the singular boundary of $\mathcal{A}$ relative to μ* is defined by

$$\partial_\mu \mathcal{A} = \{\sigma \in \overline{\mathcal{A}} : \mu(U \cap \mathcal{A}) = \infty, \quad \text{for every neighborhood } U \text{ of } \sigma\}. \tag{5.106}$$

(ii) Let $\mathcal{A}$ be a Borel subset of $\partial\Omega$. A boundary point σ is *q-accumulation point of $\mathcal{A}$* if, for every relatively open neighborhood U of σ, $C_{2/q,q'}(\mathcal{A} \cap U) > 0$. The set of q-accumulation points of $\mathcal{A}$ will be denoted by $\mathcal{A}_q^*$.

The following result, announced (under a slighly different form) in [69], is proven in [71] (see also [38], [39]).

Theorem 5.23 *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, $q \geq (n+1)/(n-1)$ and $\nu \approx (\mathcal{S}, \mu)$ an element of $\mathfrak{B}_+^{reg}(\partial\Omega)$. Then Problem (5.101) admits a solution if and only if the following condition is fulfilled :*

$$\begin{aligned} &(i) \text{ For every Borel subset } \mathcal{A} \subset \mathcal{R} = \partial\Omega \setminus \mathcal{S},\ C_{2/q,q'}(\mathcal{A}) = 0 \Longrightarrow \mu(\mathcal{A}) = 0, \\ &(ii)\, \mathcal{S} = \mathcal{S}_q^* \cup \partial_\nu(\mathcal{R}). \end{aligned} \tag{5.107}$$

One of the most striking aspect of the super-critical case is the loss of uniqueness. It has been proven by Le Gall [64] in the case $q = 2$ and extended by Marcus and Véron [71] that there exist infinitely many solutions of Problem (5.101) whenever the singular set $\mathcal{S}$ has a non-empty relative interior. Actually there exists a maximal solution, but no minimal solution. This fact has led Dynkin and Kuznetsov in [40] to introduce a thiner notion of boundary trace called the *fine trace*. However their definition is only working when $q \leq 2$. When $q = 2$ and with a fundamental use of probability techniques (the Brownian snake), Mselati proved in [80] the one to one correpondence between positive solutions of (5.86) and the fine trace. The extension of this result in the general case remains open.

5.4 General nonlinearities

5.4.1 The exponential

There are many extensions of the nonlinear boundary value problems when the nonlinearity in no longer of a power type. In [49] the boundary trace of the prescribed Gaussian curvature equation is studied

$$-\Delta u = K(x)e^{2u}, \tag{5.108}$$

in a 2-dimensional bounded domain Ω. In this equation, K is a given function ; the question is to find out a new metric conformal to the standard metric of a subdomain on the hyperbolic plane $\mathbb{H}^2$ so that K is the Gaussian curvature of this metric (see [87] for example). The existence of boundary trace in the set of outer regular Borel measures on $\partial\Omega$ is proven. In the case of a Radon measure the following existence result is obtained :

Theorem 5.24 *Suppose $\beta \leq K(x) \leq \alpha < 0$ is a continuous function in a smooth bounded domain Ω of the plane and $\mu \in \mathfrak{M}(\partial\Omega)$ with Lebesgue decomposition*

$$\mu = \mu_R dH_1 + \mu_s,$$

where $\mu_R \in L^1(\partial\Omega)$ and $\mu_s \perp \mu_R$. If there exists some $p \in (1, \infty]$ such that

$$\begin{aligned} &(i) \ \exp\left(2P_{-\Delta}^{\Omega}(\mu_s)\right) \in L^{p'}(\Omega; \rho_{\partial\Omega}dx), \\ &(ii) \ \exp(2\mu_R) \in L^{p-1}(\partial\Omega), \end{aligned} \tag{5.109}$$

then there exists a unique $u \in L^1(\Omega)$ *with* $e^{2u} \in L^1(\Omega; \rho_{\partial\Omega} dx)$ *solution of*

$$\begin{aligned} -\Delta u - K(x)e^{2u} &= 0 \quad \text{in } \Omega, \\ u &= \mu. \end{aligned} \tag{5.110}$$

As for the power case, sufficient conditions for solving

$$\begin{aligned} -\Delta u - K(x)e^{2u} &= 0 \quad \text{in } \Omega, \\ Tr_{\partial\Omega}(u) &= \nu. \end{aligned} \tag{5.111}$$

where $\nu \in \mathbb{B}_+^{reg}(\partial\Omega)$ are given. They are expressed in terms of a boundary logarithmic capacity.

5.4.2 The case of a general nonlinearity

For general semilinear equations of the form

$$-\Delta u + g(x, u) = 0 \quad \text{in } \Omega, \tag{5.112}$$

where Ω is a smooth domain in $\mathbb{R}^n$, not necessarily bounded, and g a continuous function defined on $\Omega \times \mathbb{R}$, a new approach of the boundary trace problem is provided by Marcus and Véron in [73]. As it has already been observed in the implication [(i) $\Longrightarrow$ (ii)] in the proof of Theorem 5.16, if u is a positive solution of (5.112) with $g(x, u) \geq 0$, and if for some $a \in \partial\Omega$ there exists $r > 0$ such that

$$\int_{B_r(a)\cap\Omega} g(x, u)\rho_{\partial\Omega} dx < \infty, \tag{5.113}$$

then $u \in L^1(B_{r'}(a) \cap \Omega)$ for any $0 < r' < r$ and there exists a positive linear functional ℓ on $C_c^\infty(\Sigma \cap B_r(a))$ such that, for any θ in this space,

$$\lim_{t\to 0} \int_{B_r(a)\cap\Sigma_t} u(x)\theta(x) dS_t = \ell(\theta). \tag{5.114}$$

This result leads to the notion of regular and singular points if it is assumed for example that g satisfies

$$g(x, r) \geq 0, \quad \forall (x, r) \in \Omega \times \mathbb{R}_+. \tag{5.115}$$

Definition 5.25 Let u be a continuous nonnegative solution of (5.112). A point $a \in \partial\Omega$ is called a *regular point* of u if there exists an open neighborhood U of a such that (5.113) holds. The set of regular points is denoted by $\mathcal{R}(u)$. It is a relatively open subset of $\partial\Omega$. Its complement, $\mathcal{S}(u) = \partial\Omega \setminus \mathcal{R}(u)$ is the *singular set* of u.

Using a partition of unity, it exists a positive Radon measure μ on $\mathcal{R}(u)$ such that

$$\lim_{t\downarrow 0} \int_{\mathcal{R}(u)_t} u(\sigma, t)\zeta_t(\sigma, t) dS_t = \int_{\mathcal{R}(u)} \zeta(\sigma) d\mu, \tag{5.116}$$

for every $\zeta \in C_c(\mathcal{R}(u))$.

Definition 5.26 A function g is a *coercive nonlinearity* in Ω if, for every compact subset $K \subset \Omega$, the set of positive solutions of (5.112) is uniformly bounded on K.

An example of coercive nonlinearity is the following :

$$g(x,r) \geq h(x)g(r), \quad \forall (x,r) \in \Omega \times \mathbb{R}_+, \tag{5.117}$$

where $h \in C(\Omega)$ is continuous and positive, and $f \in C(\mathbb{R}_+)$ is nondecreasing and satisfies the Keller-Osserman assumption :

$$\int_\theta^\infty \left(\int_0^t f(s)ds \right)^{-1/2} dt < \infty, \quad \forall \theta > 0. \tag{5.118}$$

The verification of this property is based upon the maximum principle and the construction of local super solutions by the Keller-Osserman method.

Definition 5.27 A function g possesses the *strong barrier property* at $a \in \partial\Omega$ if there exists $r_0 > 0$ such that, for any $0 < r \leq r_0$, there exists a positive super solution $v = v_{a,r}$ of (5.112) in $B_r(a) \cap \Omega$ such that $v \in C(B_r(a) \cap \overline{\Omega})$ and

$$\lim_{\substack{y \to x \\ y \in \Omega}} v(y) = \infty, \quad \forall x \in \Omega \times \partial B_r(a). \tag{5.119}$$

If $g(x,r) = f(r)$ where f satisfies the Keller-Osserman assumption, then it possesses the strong barrier property at any boundary point. If

$$g(x,r) = (\rho_{\partial\Omega}(x))^\alpha r^q, \quad \forall (x,r) \in \Omega \times \mathbb{R}_+$$

for some $\alpha > -2$ and $q > 1$, it possesses also the strong barrier property, but the proof, due to Du and Guo [33], is difficult in the case $\alpha > 0$ (the nonlinearity is degenerate at the boundary).

Proposition 5.28 *Let $u \in C(\Omega)$ be a positive solution of (5.112) and suppose that $a \in \mathcal{S}(u)$. Suppose that at least one of the following sets of conditions holds :*

(i) There exists an open neighborhood U' of a such that $u \in L^1(U' \cap \Omega)$.

(ii) (a) $g(x,\cdot)$ is non-decreasing in $\mathbb{R}_+$, for every $x \in \Omega$;

(b) $\exists U_a$, an open neighborhood of a, such that g is coercive in $U_a \cap \Omega$;

(c) g possesses the strong barrier property at a.

Then, for every open neighborhood U of a,

$$\lim_{t \to 0} \int_{U \cap \Sigma_t} u(x)\, dS_t = \infty. \tag{5.120}$$

This result, jointly with (5.114), yields to the following trace theorem.

Theorem 5.29 *Let g be a coercive nonlinearity which has the strong barrier property at any boundary point. Assume also that $r \mapsto g(x,r)$ is nondecreasing on $\mathbb{R}_+$ for every $x \in \Omega$. Then any continuous nonnegative solution u of (5.112) possesses a boundary trace ν in $\mathfrak{B}_+^{reg}(\partial\Omega)$ with*

$$\nu = Tr_{\partial\Omega}(u) \approx (\mathcal{S}(u), \mu), \text{ where } \mu \in \mathfrak{M}_+(\mathcal{R}(u)). \tag{5.121}$$

This result applies in the particular case where $g(x,r) = \rho_{\partial\Omega}(x)^{\alpha} r^q$. Moreover a complete extension of Theorem 5.21 in the subcritical range

$$1 < q < \frac{n+1+\alpha}{n-1}, \quad \alpha > -2,$$

is valid. The super critical case is still completely open.

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